High Dimensional Statistical Estimation under Uniformly Dithered One-bit Quantization^{*}

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Abstract

In this paper, we propose a uniformly dithered 1-bit quantization scheme for highdimensional statistical estimation. The scheme contains truncation, dithering, and quantization as typical steps. As canonical examples, the quantization scheme is applied to the estimation problems of sparse covariance matrix estimation, sparse linear regression (i.e., compressed sensing), and matrix completion. We study both sub-Gaussian and heavy-tailed regimes, where the underlying distribution of heavy-tailed data is assumed to have bounded moments of some order. We propose new estimators based on 1-bit quantized data. In sub-Gaussian regime, our estimators achieve minimax rates up to logarithmic factors, indicating that our quantization scheme costs very little. In heavytailed regime, while the rates of our estimators become essentially slower, these results are either the first ones in an 1-bit quantized and heavy-tailed setting, or already improve on existing comparable results from some respect. Under the observations in our setting, the rates are almost tight in compressed sensing and matrix completion. Our 1-bit compressed sensing results feature general sensing vector that is sub-Gaussian or even heavy-tailed. We also first investigate a novel setting where both the covariate and response are quantized. In addition, our approach to 1-bit matrix completion does not rely on likelihood and represent the first method robust to pre-quantization noise with unknown distribution. Experimental results on synthetic data are presented to support our theoretical analysis.

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1 Introduction

1-bit quantization of signals or data recently has received much attention in both signal processing and machine learning communities. In some signal processing problems, power consumption, manufacturing cost and chip area of analog-to-digital devices grow exponentially with their resolution [59]. Thus, it is impractical and infeasible to use high-precision data or signals. Alternatively, it was proposed to use low-resolution, specifically 1-bit quantization, see for instance [4, 33, 36, 58, 69, 83]. Note that, generally speaking, the quantization itself that maps an analog signal into digital representation of a finite dictionary is an inevitable process in digital signal processing [50, 51]. Besides, in many distributed machine learning or federated learning scenarios, multiple parties transmit information among themselves. The communication cost can be prohibitive for distributed algorithms where each party only has a low-power and low-bandwidth device such as a mobile device [65]. To address the bottleneck of communication cost, recent works have studied how to send a small number or even one bit per entry for such distributed machine learning applications [3, 8, 85, 92].

Because of the pratical interest of 1-bit quantization in many applications, recent years have witnessed increasing literature on high-dimensional statistical estimation from merely binary (1-bit) data, which we sometimes refer to as 1-bit estimation. Existing works tried to understand the interplay between recovery procedures and 1-bit quantization in some prototypical estimation problems, including compressed sensing¹ (e.g., [37, 40, 55, 74, 76, 89]) and matrix completion (e.g., [10,23,34,61,67]). The main goal in this area is to design quantization scheme for observed data, and of course, one hopes that accurate estimate can be obtained from the quantized data produced by the scheme. Very recently, covariance matrix estimation was also studied under 1-bit quantization by Dirksen et al. [38]; they proposed to collect 2 bits per entry for each sample by using a dithered 1-bit quantizer, and also, they developed a covariance matrix estimator that enjoys near optimal operator norm error.

We believe, however, these theoretical results are still highly insufficient and the current understanding on 1-bit estimation remains incomplete. For instance, almost all existing results heavily rely on sub-Gaussianity of the underlying distribution. While many modern datasets exhibit heavy-tailed behaviour, the 1-bit quantization of heavy-tailed data is yet to investigate. Besides, a limitation of prior results for 1-bit matrix completion is that they cannot tolerate unknown pre-quantization noise, as they require construction of likelihood. We also point out that the recent results for 1-bit covariance estimation in [40] are restricted to the lowdimensional regime.

The main goal of this paper is to promote the understanding on 1-bit estimation. Specifically, we study three fundamental high-dimensional statistical estimation problems based on data that are quantized to one bit. The quantization scheme include the typical steps of truncation, dithering, and quantization (note that truncation is for heavy-tailed data only), see Section 1.2 for detailed discussions. We present extensive theoretical results on sparse covariance matrix estimation, sparse linear regression, and low-rank matrix completion, under both sub-Gaussian data and heavy-tailed data. Here, the underlying distribution of heavy-tailed data is only assumed to have bounded moments of some order, as opposed to the conventional sub-Gaussian assumption. Our estimators in sub-Gaussian regime have remarkable statistical properties, i.e., they achieve near minimax rates (up to some logarithmic factors). In the heavy-

¹This is also referred to as sparse linear regression in statistics. In this work we will adopt more statistical conventions — we term sensing vector and (compressive) measurement as covariate and response, respectively.

tailed regime, our estimators can still deliver a faithful estimation under a high-dimensional scaling; while the error rates are essentially slower than the minimax ones because of a biasand-variance trade-off. However, to our best knowledge, these are the first high-dimensional statistical results under such two-fold predicament, i.e., heavy-tailed distribution that breaks the robustness, and 1-bit quantization that loses data information. For compressed sensing and matrix completion, we prove that the rates are nearly tight if the data are quantized under the proposed scheme and selected parameters. Here we summarize our key results and contributions as follows (For simplicity we only consider parameters n, d, s (or r), q and omit the others).

• In Section 2, for some zero-mean d-dimensional random vector X, we study the problem of estimating its covariance matrix $\Sigma^* = \mathbb{E}(XX^T) = [\sigma_{ij}^*]$, where Σ^* has the approximate column-wise sparsity structure, i.e., $\sup_{j \in [d]} \sum_{i=1}^d |\sigma_{ij}^*|^q \leq s$ for some $0 \leq q < 1$ and s > 0. Denote the full data that are i.i.d. copies of X by $X_1, ..., X_n$. For sub-Gaussian X, we i.i.d. sample the dithering noise vector $\{\Gamma_{k1}, \Gamma_{k2} : k \in [n]\}$ that are uniformly distributed on $[-\gamma, \gamma]^d$, and then dither and quantize each X_k to binary data $\operatorname{sign}(X_k + \Gamma_{k1})$, $\operatorname{sign}(X_k + \Gamma_{k2})$. Based on these binary data, we propose a thresholding estimator $\widehat{\Sigma}$, see (2.4) and (2.18). Although only two bits are collected per entry, we show a near optimal minimax rate

$$\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim s \log n \Big(\frac{\log d}{n}\Big)^{(1-q)/2}$$

For heavy-tailed X assumed to have bounded fourth moment, we first element-wisely truncate the full sample X_k to be $\widetilde{X}_k := \operatorname{sign}(X_k) \min\{|X_k|, \eta\}$ (element-wise operation). Then similar to sub-Gaussian data, we deal with \widetilde{X}_k by dithering and quantization. Our estimator possesses an estimation error bound for operator norm error

$$\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim s \Big(\frac{\log d}{n}\Big)^{(1-q)/4}$$

• In Section 3, we study sparse linear regression $Y_k = X_k^T \Theta^* + \epsilon_k$, $k \in [n]$ where the desired signal $\Theta^* = [\theta_i^*] \in \mathbb{R}^d$ satisfies $\sum_{i=1}^d |\theta_i^*|^q \leq s$ for some $0 \leq q < 1$ and s > 0, the covariate X_k and the additive noise ϵ_k can be either sub-Gaussian or heavy-tailed. Given the full data $\{(X_k, Y_k) : k \in [n]\}$, we first study a novel setting where both X_k and Y_k are quantized to binary data². The covariate X_k is quantized by exactly the same method as Section 2. For sub-Gaussian X_k and ϵ_k , the response Y_k is quantized to be $\operatorname{sign}(Y_k + \Lambda_k)$ with Λ_k uniformly distributed on $[-\gamma, \gamma]$. When X_k and ϵ_k are heavy-tailed (with bounded fourth moment), we truncate Y_k to be \widetilde{Y}_k and then similarly apply the dithered quantization to \widetilde{Y}_k . The estimation relies on the 1-bit sparse covariance matrix estimator $\widehat{\Sigma}$ developed in Section 2. To deal with the lack of positive semi-definiteness, we assume $\Sigma_{XX} = \mathbb{E}X_k X_k^T$ has column-wise sparsity, which accommodates the conventional isotropic condition (i.e., $\Sigma_{XX} = \mathbf{I}_d$) used in compressed sensing. We formulate the recovery as a convex programming problem with objective function combining a generalized quadratic loss and an ℓ_1 regularizer, see (3.17). In sub-Gaussian case, we

²The novelty here is that the covariate X_k is quantized, while all prior works on quantized compressed sensing only considered the quantization of Y_k (we refer to the survey [37]).

show our estimator $\widehat{\Theta}$ could achieve a near optimal minimax rate of

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \Big(\log n \sqrt{\frac{\log d}{n}}\Big)^{1-q/2}.$$

In heavy-tailed case, our estimator possesses the error rate

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \left(\frac{\log d}{n}\right)^{(1-q/2)/4}$$

Besides the first results for this new setting, we also revisit the canonical 1-bit compressed sensing problem where we quantize Y_k in a same manner but have full knowledge of X_k . We estimate Θ^* via analogous convex programming problems, see (3.24) and (3.28). In sub-Gaussian regime, our estimator achieves a near optimal minimax rate

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \left(\sqrt{\frac{\log d \log n}{n}}\right)^{1-q/2}$$

In heavy-tailed regime, our estimator still enjoys an error bound

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \left(\frac{\log d}{n}\right)^{(1-q/2)/3}$$

which is almost tight for the specific estimation problem where the 1-bit observation is produced by our scheme with the specified parameters (Theorem 11). As it turns out, these two results embrace some improvements on existing ones (e.g., recovery via convex programming, faster error rate), see a detailed comparison in Appendix D.

• In Section 4, we study the problem of low-rank matrix completion $Y_k = \langle \mathbf{X}_k, \Theta^* \rangle + \epsilon_k$, where the desired $d \times d$ matrix Θ^* with singular values $\sigma_1(\Theta^*) \geq ... \geq \sigma_d(\Theta^*)$ is (approximately) low-rank $\sum_{k=1}^d \sigma_k(\Theta^*)^q \leq r$ for some $0 \leq q < 1$ and r > 0. The covariate \mathbf{X}_k is uniformly distributed on $\{e_i e_j^T : i, j \in [d]\}$ where e_i is the *i*-th column of the \mathbf{I}_d , ϵ_k is sub-Gaussian or heavy-tailed noise. Given the full data $\{(\mathbf{X}_k, Y_k)\}$, we quantize Y_k to one bit by the same process as 1-bit compressed sensing in Section 3. Our estimator $\widehat{\Theta}$ is given by minimizing an objective functions constituted of a generalized quadratic loss and a nuclear norm penalty, see (4.6). If ϵ_k is sub-Gaussian, we show that $\widehat{\Theta}$ achieves a near optimal minimax rate

$$\frac{\|\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}^*\|_{\mathrm{F}}^2}{d^2} \lesssim r d^{-q} \Big(\log n \frac{d \log d}{n}\Big)^{1-q/2},$$

If ϵ_k is heavy-tailed with bounded second moment, we show the recovery guarantee

$$\frac{\|\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}^*\|_{\mathrm{F}}^2}{d^2} \lesssim r d^{-q} \Big(\frac{d \log d}{n}\Big)^{1/2 - q/4},$$

which is almost tight if one only has access to the 1-bit observation produced by our quantization scheme with the chosen parameters (Theorem 14). We emphasize that our approach is totally different from the existing method for 1-bit matrix completion, i.e., based on maximizing a likelihood function. Our essential advantage is that our method can handle unknown pre-quantization random noise that can even be heavy-tailed. See more discussions in Appendix D.

The rest of the paper is structured as follows. In the remainder of Section 1 we introduce the notation, propose the 1-bit quantization scheme, and provide some intuitions for our results (i.e., near optimality in sub-Gaussian case, and the rate degradation in heavy-tailed case). Our main results for three estimation problems are presented in Sections 2-4; In Section 5 we provide an overview of the proofs and the main techniques; In Section 6, we present experimental results to corroborate our theories; We finally give some concluding remarks in Section 7. The complete proofs are provided in Appendices A-C. Also, a comparison between this work and the literature is given in Appendix D (review of related work is provided there to avoid a lengthy beginning), while details of the numerical simulations are deferred to Appendix E.

1.1 Notations and Preliminaries

As general principles, lowercase letters (e.g., s, r) represent scalars, capital letters (e.g., X, Y) represent vectors, and capital bold letters (e.g., X, Θ) represent matrices. Some exceptions are that we use capital letter Y, Y_k to denote the responses, Λ, Λ_k to denote the dithering noise for Y, Y_k , and $X_{k,i}$ for the i-th entry of X_k . Notations marked by * denote the desired underlying signals, e.g., $\Sigma^*, \Theta^*, \Theta^*$, while those with a hat denote our estimators, e.g., $\hat{\Sigma}, \hat{\Theta}, \hat{\Theta}$.

We first introduce different vector or matrix norms. Let $[N] = \{1, 2, ..., N\}$. For a vector $X = [x_i] \in \mathbb{R}^d$, the ℓ_1 norm, ℓ_2 norm and max norm are given by $\|X\|_1 = \sum_{i=1}^d |x_i|, \|X\|_2 = (\sum_{i=1}^d |x_i|^2)^{1/2}, \|X\|_{\max} = \max_{i \in [d]} |x_i|$, respectively. Note that we also use $\|X\|_0$ to denote the number of non-zero entries in X. For a matrix $\boldsymbol{X} = [x_{ij}] \in \mathbb{R}^{d \times d}$, the operator norm, Frobenius norm and max norm are defined as $\|\boldsymbol{X}\|_{op} = \sup_{\|V\|_2=1} \|\boldsymbol{X}V\|_2, \|\boldsymbol{X}\|_{\mathrm{F}} = (\sum_{i=1}^d \sum_{j=1}^d x_{ij}^2)^{1/2}, \|X\|_{\max} = \max_{1 \le i,j \le d} |x_{ij}|$. Assume the singular values are $\sigma_1(\boldsymbol{X}) \ge \sigma_2(\boldsymbol{X}) \ge ... \ge \sigma_d(\boldsymbol{X})$, then the nuclear norm $\|\boldsymbol{X}\|_{\mathrm{nu}} = \sum_{i=1}^d \sigma_i(\boldsymbol{X})$ serves as the counterpart of the ℓ_1 norm of vectors. Given $\boldsymbol{A} = [\alpha_1, ..., \alpha_d] \in \mathbb{R}^{d \times d}$, we use vec(\cdot) to vectorize \boldsymbol{A} in a column-wise manner, i.e., vec(\boldsymbol{A}) = $[\alpha_1^T, \alpha_2^T, ..., \alpha_d^T]^T$, while the inverse of vec(\cdot) is denoted by mat(\cdot). Assume $\boldsymbol{B} \in \mathbb{R}^{d \times d}$, then the inner product in $\mathbb{R}^{d \times d}$ is defined as $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{Tr}(\boldsymbol{A}^T \boldsymbol{B}) = \operatorname{vec}(\boldsymbol{A})^T \operatorname{vec}(\boldsymbol{B})$.

Throughout the paper, we use n to denote the number of samples in data, while d the ambient dimension of the problem. Expectation and probability are denoted by $\mathbb{E}(\cdot)$, $\mathbb{P}(\cdot)$ respectively. For a specific event E, $\mathbb{1}(E)$ stands for the corresponding indicator function, i.e., $\mathbb{1}(E) = 1$ if E happens, $\mathbb{1}(E) = 0$ otherwise. We work with quite a lot of parameters arising in several signal processing steps. To avoid confusion of constants, we use $\{D_1, D_2, D_3, ...\}$ to denote constants whose values may vary from line to line, while $\{C_1, C_2, C_3, ...\}$ would only be used once to set a specific parameter, see (2.6), (2.9) for example.

We adopt standard asymptotic notations that omits absolute constants. Specifically, we use $B_1 \leq B_2$ or $B_1 = O(B_2)$ to abbreviate the fact that $B_1 \leq CB_2$ for some absolute constant C. Similarly, we write $B_1 \gtrsim B_2$ or alternatively $B_1 = \Omega(B_2)$ if $B_1 \geq CB_2$ for some C > 0. If both $B_1 = O(B_2)$ and $B_1 = \Omega(B_2)$ hold, i.e., B_1 equals B_2 up to constants, we write $B_1 \simeq B_2$.

The function $\operatorname{sign}(\cdot)$ extracts the sign of a real number x, i.e., $\operatorname{sign}(x) = 1$ if $x \ge 0$, $\operatorname{sign}(x) = -1$ if x < 0. Hard thresholding operator with threshold ζ is defined by $\mathcal{T}_{\zeta}(x) = x\mathbb{1}(|x| \ge \zeta)$. Both $\operatorname{sign}(\cdot)$ and $\mathcal{T}_{\zeta}(\cdot)$ operate on vectors or matrices element-wisely.

To broaden the range of our readers, we give some preliminaries on sub-Gaussian random variable or concentration inequality as follows.

Definition 1. Given a real random variable $X \in \mathbb{R}$, its sub-Gaussian norm $||X||_{\psi_2}$, sub-

exponential norm $||X||_{\psi_1}$ are defined as

$$\|X\|_{\psi_2} = \inf\left\{t > 0 : \mathbb{E}\exp\left(\frac{X^2}{t^2}\right) \le 2\right\} \quad , \quad \|X\|_{\psi_1} = \inf\left\{t > 0 : \mathbb{E}\exp\left(\frac{|X|}{t}\right) \le 2\right\}.$$
(1.1)

X is said to be sub-Gaussian if $||X||_{\psi_2} \leq \infty$.

Definition 2. Given a real random vector $X \in \mathbb{R}^d$, the sub-Gaussian norm is defined as $\|X\|_{\psi_2} = \sup_{\|V\|_2=1} \|V^T X\|_{\psi_2}$. X is said to be sub-Gaussian if $\|X\|_{\psi_2} \leq \infty$.

For $X, Y \in \mathbb{R}$ we note a useful relation (see [93, Lemma 2.7.7])

$$\|XY\|_{\psi_2} \le \|X\|_{\psi_1} \|Y\|_{\psi_1}.$$
(1.2)

Sub-Gaussian variable X has properties similar to the Gaussian one, such as light probability tail and bounded moment constraint.

Proposition 1. (Proposition 2.5.2, [93]). Assume random variable X is sub-Gaussian, then for absolute constants D_1, D_2 we have:

- (a) For any t > 0, $\mathbb{P}(|X| \ge t) \le 2 \exp(-\frac{D_1 t^2}{\|X\|_{w_0}^2})$.
- (b) For any $p \ge 1$, $(\mathbb{E}|X|^p)^{1/p} \le D_2 ||X||_{\psi_2} \sqrt{p}$.

Proposition 2. (Proposition 2.6.1, [93]). Let $X_1, ..., X_N$ be independent, zero-mean, sub-Gaussian random variables, then for some absolute constant D_1 we have $\left\|\sum_{k=1}^N X_k\right\|_{\psi_2}^2 \leq D_1 \sum_{k=1}^N \|X_k\|_{\psi_2}^2$.

For concentration results, we only introduce Hoeffding's inequality and Bernstein's inequality. Several other concentration inequalities (e.g., Matrix Bernstein's inequality) would be properly referred to the sources when they are invoked in the proof.

Proposition 3. (Hoeffding's inequality, [82, Theorem 1.9]). Let $X_1, ..., X_n$ be independent, bounded random variables satisfying $X_i \in [a_i, b_i]$, then for any t > 0 it holds that

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n}(X_{k}-\mathbb{E}X_{k})\Big| \ge t\Big) \le 2\exp\Big(-\frac{2n^{2}t^{2}}{\sum_{i=1}^{n}(b_{i}-a_{i})^{2}}\Big).$$
(1.3)

Proposition 4. (Bernstein's inequality, [93, Theorem 2.8.1]). Let $X_1, ..., X_N$ be independent random variables, then for any t > 0 and for some absolute constant D_1 we have

$$\mathbb{P}\Big(\Big|\sum_{k=1}^{N} (X_k - \mathbb{E}X_k)\Big| \ge t\Big) \le 2\exp\Big(-D_1\min\Big\{\frac{t^2}{\sum_{k=1}^{N} \|X_k\|_{\psi_1}^2}, \frac{t}{\max_{k\in[N]} \|X_k\|_{\psi_1}}\Big\}\Big).$$
(1.4)

Although sub-Gaussian data has exciting statistical properties like similar tail bounds as Gaussian distribution, data in some real problems may have much heavier tail, to name a few, data in economics and finance [54], biomedical data [13,96], noise in signal processing [88,95], and even signal itself [2, 66, 68]. Therefore, we will also consider the 1-bit quantization of heavy-tailed data. We use bounded moment of some order to capture the heavy-tailedness, i.e., $\mathbb{E}|X|^l \leq M$ for some l > 0. Note that this is a widely used definition [44,45,53,57,87,95,99].

1.2 1-bit Quantization Scheme

Truncation, dithering and quantization are three typical signal processing steps in our work. We summarize our 1-bit quantization scheme as follows:

- 1. **Truncation.** The truncation step will only be used to heavy-tailed data. Specifically, we first specify a threshold $\eta > 0$, then the truncation step shrinks a scalar x to be $\operatorname{sign}(x) \min\{|x|, \eta\}$, and hence x with magnitude smaller than η remains unchanged in truncation. Vectors are truncated element-wisely. Notations marked by tilde are used exclusively to denote truncated data, for example, \widetilde{X}_k and \widetilde{Y}_k .
- 2. Dithering. The dithering step is applied to all the data that we plan to quantize to 1 bit. For $E \subset \mathbb{R}^m$, we use $X \sim \operatorname{uni}(E)$ to state that X obeys uniform distribution on E. In sub-Gaussian case we dither the covariate X_k and response Y_k by uniformly distributed noise. Note that we need to sample two bits per entry for X_k (the covariate in sparse linear regression). Thus, we draw $\Gamma_{k1}, \Gamma_{k2} \sim \operatorname{uni}([-\gamma, \gamma]^d)$ and dither X_k to be $X_k + \Gamma_{k1}, X_k + \Gamma_{k2}$. We only need 1-bit information for each response Y_k , so we sample $\Lambda_k \sim \operatorname{uni}([-\gamma, \gamma])$ and obtain the dithered response $Y_k + \Lambda_k$. In heavy-tailed case X_k and Y_k are substituted with the truncated data \widetilde{X}_k and \widetilde{Y}_k .
- 3. Quantization. In quantization step we simply apply $\operatorname{sign}(\cdot)$ to the dithered data, and notations marked by a dot (e.g., $\dot{Y}_k, \dot{X}_{k1}, \dot{X}_{k2}$) exclusively represent the 1-bit quantized data. More precisely, we have $\dot{Y}_k = \operatorname{sign}(Y_k + \Lambda_k), \ \dot{X}_{kj} = \operatorname{sign}(X_k + \Gamma_{kj}), \ j = 1, 2$ for sub-Gaussian $X_k, \ Y_k$, and $\dot{Y}_k = \operatorname{sign}(\widetilde{Y}_k + \Lambda_k), \ \dot{X}_{kj} = \operatorname{sign}(\widetilde{X}_k + \Gamma_{kj}), \ j = 1, 2$ for heavy-tailed X and Y.

1.3 Intuition and Heuristic

Here we illustrate the intuition of our results before proceeding to details. Readers familiar with the 1-bit quantization with uniform dithering noise can directly skip this part.

Specifically, we will heuristically analyse a multi-bit matrix completion setting to illustrate the the reason why our estimators could achieve near optimal minimax rates in sub-Gaussian regime. In fact, the idea of the whole paper stems from two simple observations, which are given in the following two lemmas. We mention that Corollary 1 motivates [38] to estimate $\mathbb{E}(XY)$ and hence an unstructured covariance matrix via binary data, while Lemma 1 is its more elementary version and enlightens the estimators in our work. For instance, while full observations are not available, our loss function in matrix completion is constructed by substituting the full data Y_k in the empirical ℓ_2 loss with the 1-bit surrogate $\gamma \cdot \dot{Y}_k$ (see (4.6)). This idea comes from Lemma 1.

Lemma 1. Let X, Λ be two independent random variables satisfying $|X| \leq B, \Lambda \sim \text{uni}([-\gamma, \gamma])$ where $\gamma \geq B$, then we have $\mathbb{E}[\gamma \cdot \text{sign}(X + \Lambda)] = \mathbb{E}X$.

Corollary 1. (Lemma 16 in [38]). Let X, Y be bounded random variables satisfying $|X| \leq B$, $|Y| \leq B$, Λ_1, Λ_2 are i.i.d. uniformly distributed on $[-\gamma, \gamma], \gamma \geq B$, and Λ_1, Λ_2 are independent of X, Y. Then we have $\mathbb{E}[\gamma^2 \cdot \operatorname{sign}(X + \Lambda_1) \cdot \operatorname{sign}(Y + \Lambda_2)] = \mathbb{E}XY$.

Next, by informal arguments, we heuristically compare full-data-based matrix completion and quantized matrix completion where one can sample finitely many bits from each Y_k (we refer it as multi-bit matrix completion). This comparison can provide some insights of why our estimators can achieve a near optimal minimax rate in sub-Gaussian regime.

We consider a full-data sample of size n from matrix completion (4.1) and denote it by

$$\mathcal{D}_{\text{full}} = \Big\{ (\boldsymbol{X_1}, Y_1), ..., (\boldsymbol{X_n}, Y_n) \Big\}.$$

For some positive integer f(n) we i.i.d. draw $\{\Lambda_{kj} : j \in [f(n)]\}$ from $\operatorname{uni}([-\gamma, \gamma])$, and sample f(n) bits from each Y_k by the proposed dithered quantization, that is, $\{\dot{Y}_{kj} := \operatorname{sign}(Y_k + \Lambda_{kj}) : j \in [f(n)]\}$. This quantization process yields the sample containing $n \cdot f(n)$ binary observations

$$\mathcal{D}_{\text{mult}} = \left\{ (\boldsymbol{X}_{\boldsymbol{k}}, \dot{Y}_{kj}) : k \in [n], \ j \in [f(n)] \right\}$$

Interestingly, from $\mathcal{D}_{\text{mult}}$ one can build a dataset with size n as

$$\mathcal{D}_{\text{appr}} = \left\{ (\boldsymbol{X}_{\boldsymbol{k}}, Y_{k, \text{appr}}) : Y_{k, \text{appr}} = \frac{1}{f(n)} \sum_{j \in [f(n)]} \gamma \cdot \dot{Y}_{kj}, \ k \in [n] \right\}$$

We aim to reveal that the above three samples are comparably informative for the estimation.

For simplicity we assume $\|\epsilon_k\|_{\psi_2} = O(1)$, $\|Y_k\|_{\psi_2} = O(1)$, then with probability at least $1 - O(n^{-\Omega(1)})$ we have $\max_k |Y_k| = O(\sqrt{\log n})$ (see [82, Theorem 1.14]). Thus, we can choose $\gamma = \operatorname{Poly}(\log n)^3$ to guarantee $\gamma > \max_k |Y_k|$ with high probability. We proceed the analysis on this event. Define $\epsilon_{k,\operatorname{appr}} := Y_{k,\operatorname{appr}} - Y_k$, equivalently we can write

$$Y_{k,\text{appr}} = Y_k + \epsilon_{k,\text{appr}} = \left\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* \right\rangle + \epsilon_k + \epsilon_{k,\text{appr}}.$$
(1.5)

For $\epsilon_{k,\text{appr}}$, Lemma 1 gives $\mathbb{E}_{\Lambda_{kj}}(\gamma \cdot \dot{Y}_{kj}) = Y_k$ and hence $\mathbb{E}\epsilon_{k,\text{appr}} = 0$. Moreover, conditioned on Y_k , $\epsilon_{k,\text{appr}}$ is the mean of f(n) zero-mean, independent random variables lying in $[-\gamma - Y_k, \gamma - Y_k]$. Thus, Proposition 2 and Hoeffding's Lemma (see Lemma 1.8, [82]) give

$$\|\epsilon_{k,\text{appr}}\|_{\psi_2} = O\left(\frac{\gamma}{\sqrt{f(n)}}\right). \tag{1.6}$$

Therefore, $\|\epsilon_{k,appr}\|_{\psi_2} = O(1)$ as long as f(n) dominates γ^2 , while $f(n) = \text{Poly}(\log n)$ would suffice due to $\gamma = \text{Poly}(\log n)$. In conclusion, $\mathcal{D}_{\text{mult}}$ containing $n \cdot \text{Poly}(\log n)$ binary data can generate the sample \mathcal{D}_{appr} of size n, where each $Y_{k,appr}$ can be viewed as a full observation from (1.5). Moreover, since $\|\epsilon_k\|_{\psi_2} = \|\epsilon_{k,appr}\|_{\psi_2} = O(1)$, (1.5) is almost equivalent to the original model (4.1). This reveals \mathcal{D}_{appr} , and hence \mathcal{D}_{appr} with $n \cdot \text{Poly}(\log n)$ binary observations, are comparable to $\mathcal{D}_{\text{full}}$ with n full observations. Furthermore, this indicates the inessential logarithmic degradation of recovery error after 1-bit quantization.

Note that similar heuristics can be found in sub-Gaussian regime of (sparse) covariance matrix estimation and sparse linear regression. Of course, such multi-bit heuristic deviates from the 1-bit setting where we collect only 1 bit from each Y_k (see the following graphical illustration). But since $f(n) := \text{Poly}(\log n)$ is negligible compared with n, one may tend to believe $\mathcal{D}_{\text{mult}}$ and $\mathcal{D}_{1\text{bit}} = \{\dot{Y}_k := \text{sign}(Y_k + \Lambda_k) : k \in [n \cdot f(n)]\}$ are comparable. From this perspective, the near-optimal rates in sub-Gaussian regime are merely matter of courses.

³Here Poly(log n) denotes any term T satisfying $T = O([\log n]^m)$ for some positive integer m.



However, in heavy-tailed regime the story becomes totally different. Specifically, $\gamma = \text{Poly}(\log n)$ will no longer guarantee $\gamma > \max_k |Y_k|$ with high probability. When this vital condition fails, the dithering becomes invalid for responses with absolute value larger than γ . Indeed, for these measurements the proposed dithered quantization reduces to a direct collection of the sign, while under such direct quantization we even lose the well-posedness of the problem (e.g., matrix completion, see [34]) or the possibility of full signal reconstruction (e.g., 1-bit compressed sensing, see [75]).

To resolve the issue, we truncate the heavy-tailed data according to some threshold η , which produces data bounded by η . Then we can treat the truncated data as sub-Gaussian data and use dithering noise drawn from $\operatorname{uni}([-\gamma, \gamma])$ with $\gamma > \eta$. It is not hard to see that η represents the data bias introduced in truncation. More precisely, smaller η corresponds to larger bias. Because of Hoeffding's Lemma, γ is positively related to data variance. Definitely, for estimation or signal recovery we prefer data with small bias (i.e., big η) and small variance (i.e., small γ). But, note that we also need $\gamma > \eta$ to enforce the effectiveness of dithering. Thus, a trade-off between bias and variance is needed. We comment that, making an optimal balance between bias and variance leads to our error rates in heavy-tailed regime. See Example 1 in Section 5 for instance.

2 Sparse Covariance Matrix Estimation

We start from the problem of estimating a sparse covariance matrix. Let $X \in \mathbb{R}^d$ be a random vector with zero mean, the i.i.d. realizations X_k are quantized to 1-bit data $(\dot{X}_{k1}, \dot{X}_{k2})$, and we aim to estimate the underlying covariance matrix $\Sigma^* = \mathbb{E}XX^T$ based on the quantized data.

We first ideally assume the underlying d-dimensional random vector X has entries bounded by γ , then Corollary 1 delivers that $\mathbb{E}\left[\gamma^2 \cdot \dot{X}_{k1} \dot{X}_{k2}^T\right] = \mathbb{E}XX^T = \Sigma^*$, which is just the desired covariance matrix. Besides, the concentration of $\gamma^2 \cdot \dot{X}_{k1} \dot{X}_{k2}^T$ should be fast due to boundedness, see Hoeffding's inequality in Proposition 3. Combining the two observations, [38] proposed a covariance matrix estimator as an empirical version of $\mathbb{E}\left[\gamma^2 \cdot \dot{X}_{k1} \dot{X}_{k2}^T\right]$, followed by symmetrization:

$$\breve{\Sigma} = \frac{\gamma^2}{2n} \sum_{k=1}^n \left[\dot{X}_{k1} \dot{X}_{k2}^T + \dot{X}_{k2} \dot{X}_{k1}^T \right].$$
(2.1)

For sub-Gaussian X_k , this estimator achieves a near minimax rate (compared with full data setting in [19])

$$\|\breve{\Sigma} - \Sigma^*\|_{\text{op}} \lesssim \log n \sqrt{\frac{d \log d}{n}}.$$
 (2.2)

Here, we point out that sampling two bits (rather than one bit) per entry is merely for estimating the diagonal entries of Σ^* , since the 1-bit version of (2.1),

$$\breve{\boldsymbol{\Sigma}}_{1\text{bit}} = \frac{\gamma^2}{n} \sum_{k=1}^n \dot{X}_{k1} \dot{X}_{k1}^T,$$

always gives γ^2 in the diagonal and hence fails to recover the diagonal of the covariance matrix.

It is evident that (2.2) requires at least $n \gtrsim d$ to provide a non-trivial error bound. Actually it has been reported that even the sample covariance matrix $\sum_k X_k X_k^T/n$ has extremely poor performance under high dimensional scaling where $d \ge n$ [56], not to mention (2.1). On the other hand, high-dimensional databases are undoubtedly becoming ubiquitous in genomics, biomedical, imaging, tomography, finance and so forth, while covariance matrix plays a fundamental role in the analysis of these databases.

To address the high-dimensional issue, extra structures are necessary to reduce the intrinsic problem dimensionality. For covariance matrix we usually have sparsity as prior knowledge, especially in the situation where dependencies among different features are weak, for instance, the Genomics data [41], functional data drawn from underlying curves [79]. A precise formulation of the sparse structure is provided in Assumption 1.

Assumption 1. (Approximate column-wise sparsity). For a specific $0 \le q < 1$, the columns of covariance matrix $\Sigma^* = [\sigma_{ii}^*]$ are approximately sparse in the sense that

$$\sup_{j \in [d]} \sum_{i=1}^{d} |\sigma_{ij}^*|^q \le s \tag{2.3}$$

In literature there are two mainstreams to incorporate sparsity into covariance matrix estimation, namely penalized likelihood method [12, 84] and a thresholding method [11, 18, 20, 22, 42]. Thresholding method refers to the direct regularizer that element-wisely hard thresholding the sample covariance matrix, i.e., $\mathcal{T}_{\zeta}(\sum_{k=1}^{n} X_k X_k^T/n)$, which promotes sparsity intuitively. With suitable threshold ζ , Cai and Zhou [21] showed $\mathcal{T}_{\zeta}(\sum_{k=1}^{n} X_k X_k^T/n)$ could achieve minimax rate under operator norm over the class of column-wisely sparse covariance matrices (Assumption 1). Motivated by previous work, we propose to hard thresholding $\check{\Sigma}$ in (2.1) to obtain a high-dimensional estimator $\hat{\Sigma} = [\hat{\sigma}_{ij}]$ given by

$$\widehat{\Sigma} = \mathcal{T}_{\zeta} \widecheck{\Sigma}. \tag{2.4}$$

The statistical rates of $\widehat{\Sigma}$ under both max norm and operator norm are established in what follows.

2.1 Sub-Gaussian Data

Assume $X_k = [X_{k,1}, X_{k,2}, ..., X_{k,d}]$ are i.i.d. sampled from a random vector $X \in \mathbb{R}^d$ with zero-mean sub-Gaussian components. In particular, we assume

$$\mathbb{E}X_{k} = 0, \ \|X_{k,i}\|_{\psi_{2}} \le \sigma, \ \forall i \in [d].$$
(2.5)

From (2.4), $\check{\Sigma} = [\check{\sigma}_{ij}]$ serves as an intermediate estimator to construct $\widehat{\Sigma}$, hence we first provide an element-wise error bound of $\check{\Sigma}$ in Theorem 1.

Theorem 1. Assume (2.5) holds. For specific $\delta \geq 1$ we assume $n > 2\delta \log d$. For some sufficiently large constant C_1 we set the dithering scale γ as

$$\gamma = C_1 \sigma \sqrt{\log\left(\frac{n}{2\delta \log d}\right)} \tag{2.6}$$

and assume $\gamma > \sigma$. Then for $\breve{\Sigma} = [\breve{\sigma}_{ij}]$ we have

$$\mathbb{P}\Big(|\breve{\sigma}_{ij} - \sigma_{ij}^*| \lesssim \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big) \ge 1 - 2d^{-\delta}$$
(2.7)

for $i, j \in [d]$. Moreover, we have the error bound for max norm

$$\mathbb{P}\Big(\|\breve{\boldsymbol{\Sigma}} - {\boldsymbol{\Sigma}}^*\|_{\max} \lesssim \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big) \ge 1 - 2d^{2-\delta}.$$
(2.8)

Recall that our estimator is obtained by hard thresholding $\check{\Sigma}$. The next Theorem shows that with suitable threshold ζ , the hard thresholding even brings a tighter statistical bound for element-wise error.

Theorem 2. Assume (2.5) holds, $\delta \geq 1$ is the same as Theorem 1, and the dithering scale γ is given as (2.6) with some C_1 . Then we choose the threshold ζ by

$$\zeta = C_2 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}},\tag{2.9}$$

where C_2 is a sufficiently large constant. Then for any $i, j \in [d]$ we have

$$\mathbb{P}\Big(|\widehat{\sigma}_{ij} - \sigma_{ij}^*| \lesssim \min\left\{|\sigma_{ij}^*|, \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right\}\Big) \ge 1 - 2d^{-\delta}.$$
(2.10)

By combining (2.10) and Assumption 1, we are in a position to establish the rate of $\widehat{\Sigma}$ under operator norm. Specifically, we prove that our 1-bit estimator achieves a rate $O(s((\log n)^2 \frac{\log d}{n})^{(1-q)/2})$, which almost matches the minimax rate $O(s(\frac{\log d}{n})^{(1-q)/2})$ proved in [21, Theorem 2]. Note that the estimator based on full data in [21] achieves the minimax rate. From this perspective, the 1-bit quantization only introduces minor information loss to the learning process, i.e., a logarithmic factor. Thus, by using our method, one can embrace the privileges of 1-bit data and accurate covariance matrix estimation simultaneously.

Theorem 3. Assume Assumption 1, (2.5) hold, δ is the same as Theorem 1, 2 (set $\delta \geq 4$), and the dithering scale γ , the threshold ζ are respectively given by (2.6), (2.9) with some C_1, C_2 . Besides, assume $\delta \log d(\log n)^2/n$ is sufficiently small. Let $p = \delta/4$, we have

$$\left(\mathbb{E}\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\mathrm{op}}^p\right)^{1/p} \lesssim s \left(\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right)^{1-q}.$$
(2.11)

Moreover, the probability tail of operator norm error is bounded as

$$\mathbb{P}\Big(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim s\Big[\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big]^{1-q}\Big) \ge 1 - \exp(-\delta).$$
(2.12)

Remark 1. We point out that the proof of Theorem 3 may be of independent technical interest, especially the probabilistic inequality (2.12) that seems quite new in the literature. In fact, only the upper bound for the second moment (i.e., p=2 and $\delta = 8$ in (2.11)) is obtained in literature (e.g., [21, Theorem 3]), and by Markov inequality this can only give a probability term $1 - \frac{1}{\delta^{1-q}}$ in (2.12). Here, by contrast, we derive a much better probabilistic term $1 - \exp(-\delta)$. The key idea is to adaptively bound the $\Omega(\delta)$ -th moment rather than a specific second moment, which gives (2.11). It is straightfoward to apply this method to the traditional full-data thrsholding estimator and gain some improvement on prior results.

To guarantee positive semi-definiteness, we introduce a trick developed in literature. Let the eigenvalue decomposition of $\widehat{\Sigma}$ be $\sum_{i=1}^{d} \lambda_i(\widehat{\Sigma}) v_i v_i^T$, we remove the components corresponding to negative eigenvalues and obtain the positive part $\widehat{\Sigma}^+ = \sum_{i=1}^{d} \max(\lambda_i(\widehat{\Sigma}), 0) v_i v_i^T$. It is not hard to show that $\|\widehat{\Sigma}^+ - \Sigma^*\|_{\text{op}} \leq 2\|\widehat{\Sigma} - \Sigma^*\|_{\text{op}}$. Thus, $\widehat{\Sigma}^+$ retains the operator norm rate of $\widehat{\Sigma}$. However, removing the negative components may destroy the element-wise error or the sparse pattern of $\widehat{\Sigma}$, see [78].

Besides, it is worth noting that we present Theorem 3 under operator norm by convention, but both (2.11) and (2.12) are applicable to the larger norm $\|\boldsymbol{X}\|_{1,\infty} = \sup_j \sum_i |x_{ij}|$, see an initial step in the proof (A.6).

2.2 Heavy-tailed Data

Let $X_k = [X_{k,1}, ..., X_{k,d}]^T$ be i.i.d. drawn from the random vector $X \in \mathbb{R}^d$, in this part we consider zero-mean, heavy-tailed X assumed to have bounded fourth moments

$$\mathbb{E}X_k = 0, \mathbb{E}|X_{k,i}|^4 \le M, \quad \forall \ i \in [d].$$

$$(2.13)$$

Note that this offers great relaxation compared to sub-Gaussian random variable and encompasses more distributions such as t-distribution, log-normal distribution.

Compared with the light tail in Proposition 1(a), X satisfying (2.13) can have a much heavier tail, and so overlarge data appear more frequently. This is problematic because our dithering noise has finite scale γ , hence the dithering is invalid for data with magnitude larger than γ . More precisely, this issue can be formulated as

$$\operatorname{sign}(X_{k,i} + \Gamma_{kj,i}) = \operatorname{sign}(X_{k,i}), \quad \text{if } |X_{k,i}| > \gamma.$$

Therefore, for those entries larger than γ , our signal processing reduces to a direct quantization without dithering noise, which is known to introduce great loss of information.

To deal with the issue, we first truncate the data larger than a specified threshold η and obtain the truncated data \tilde{X}_k bounded by η , which is of the spirit to introduce some biases for variance reduction. Now that the truncated data are bounded, we similarly dither them by uniform noise, and then quantize to $\dot{X}_{kj} = \text{sign}(\tilde{X}_{kj} + \Gamma_{kj}), j = 1, 2$, where $\Gamma_{kj} \sim \text{uni}([-\gamma, \gamma]^d)$. Motivated by Corollary 1, we propose an intermediate estimator

$$\breve{\Sigma} = \frac{\gamma^2}{2n} \sum_{k=1}^n \left[\dot{X}_{k1} \dot{X}_{k2}^T + \dot{X}_{k2} \dot{X}_{k1}^T \right], \qquad (2.14)$$

which extends (2.1) to heavy-tailed data. Element-wise error for $\check{\Sigma}$ is given in Theorem 4.

Theorem 4. Assume (2.13) holds. For some fixed $\delta \geq 1$ and C_3, C_4 ($C_4 > C_3$), we set the truncation parameter η and the dithering scale γ by

$$\begin{cases} \eta = C_3 M^{1/4} \left(\frac{n}{\delta \log d}\right)^{1/8} \\ \gamma = C_4 M^{1/4} \left(\frac{n}{\delta \log d}\right)^{1/8} \end{cases}, \tag{2.15}$$

Then for $\breve{\Sigma} = [\breve{\sigma}_{ij}]$ given in (2.14), we have

$$\mathbb{P}\left(\left|\breve{\sigma}_{ij} - \sigma_{ij}^*\right| \lesssim \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}\right) \ge 1 - 2d^{\delta}.$$
(2.16)

Moreover, we have the error bound under max norm

$$\mathbb{P}\Big(\|\breve{\boldsymbol{\Sigma}} - {\boldsymbol{\Sigma}}^*\|_{\max} \lesssim \sqrt{M} \Big[\frac{\delta \log d}{n}\Big]^{1/4}\Big) \ge 1 - 2d^{2-\delta}.$$
(2.17)

Parallel to the sub-Gaussian regime, we use an additional hard thresholding step to promote sparsity. That is, based on the intermediate estimator $\check{\Sigma}$ in (2.14), we choose some suitable thresholding parameter ζ and define the estimator

$$\widehat{\Sigma} = \mathcal{T}_{\zeta} \widecheck{\Sigma}. \tag{2.18}$$

We show the element-wise and operator norm statistical rates in Theorem 5, Theorem 6.

Theorem 5. Assume (2.13) holds, δ is the same as Theorem 4, and the truncation threshold η and the dithering scale γ are set as (2.15) with some C_3, C_4 . Then we set the threshold ζ in (2.18) by

$$\zeta = C_5 \sqrt{M} \left(\frac{\delta \log d}{n}\right)^{1/4} \tag{2.19}$$

where C_5 is a sufficiently large constant. Then for any $i, j \in [d]$ we have

$$\mathbb{P}\Big(|\widehat{\sigma}_{ij} - \sigma_{ij}^*| \lesssim \min\left\{|\sigma_{ij}^*|, \sqrt{M} \Big[\frac{\delta \log d}{n}\Big]^{1/4}\right\}\Big) \ge 1 - 2d^{-\delta}.$$
(2.20)

Theorem 6. Assume Assumption 1, (2.13) hold, δ is fixed and the same as Theorem 4, 5 (set $\delta \geq 4$), the truncation threshold η , the dithering scale γ , the threshold ζ are set as (2.15), (2.19) for some specified C_3, C_4, C_5 . Besides, assume that $\delta \log d/n$ is sufficiently small. Let $p = \delta/4$, then we have the bound for the moment of order p

$$\left(\mathbb{E}\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\rm op}^p\right)^{1/p} \lesssim sM^{(1-q)/2} \left[\frac{\delta \log d}{n}\right]^{(1-q)/4}.$$
(2.21)

Moreover, we bound the probability tail of operator norm error

$$\mathbb{P}\Big(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim sM^{(1-q)/2} \Big[\frac{\delta \log d}{n}\Big]^{(1-q)/4}\Big) \ge 1 - \exp(-\delta).$$
(2.22)

3 Sparse Linear Regression

We intend to establish our results for sparse linear regression (Section 3) and low-rank matrix completion (Section 4) under the unified framework of trace regression, which should be established first. Trace regression with $\Theta^* \in \mathbb{R}^{d \times d}$ as desired signal is formulated as

$$Y_k = \left\langle \boldsymbol{X_k}, \boldsymbol{\Theta^*} \right\rangle + \epsilon_k, \tag{3.1}$$

where $X_k \in \mathbb{R}^{d \times d}$ is covariate, ϵ_k is additive noise. To handle high-dimensional scaling, Θ^* is assumed to be (approximately) low-rank (e.g., [45, 70, 71])

$$\sum_{k=1}^{d} |\sigma_k(\boldsymbol{\Theta}^*)|^q \le r, \text{ for some } 0 \le q < 1,$$
(3.2)

where $\sigma_1(\Theta^*) \ge \sigma_2(\Theta^*) \ge ... \ge \sigma_d(\Theta^*)$ are the singular values of Θ^* . For this low-rank trace regression problem, a standard approach to estimate or reconstruct Θ^* is via the M-estimator (e.g., [72])

$$\widehat{\boldsymbol{\Theta}} \in \underset{\boldsymbol{\Theta} \in \mathcal{S}}{\operatorname{arg\,min}} \ \mathcal{L}(\boldsymbol{\Theta}) + \lambda \|\boldsymbol{\Theta}\|_{\operatorname{nu}}, \tag{3.3}$$

where $\mathcal{L}(\Theta)$ is a loss function that requires Θ to fit the data $\{(X_k, Y_k)\}, \|\Theta\|_{nu}$ is the penalty that promotes low-rankness. In [70] Negahban and Wainwright first established a general framework to obtain convergence rate for trace regression when $\mathcal{L}(\Theta)$ is a quadratic loss, and then many subsequent papers developed and extended the theoretical framework, to name a few, negative log-likelihood loss function [43], other estimation problems such as matrix completion with sparse corruption [62] and sparse high-dimensional time series [6], extension to quaternion field [28]. For data fitting term $\mathcal{L}(\Theta)$, a standard quadratic loss (i.e., ℓ_2 loss) based on full data is

$$\mathcal{L}(\boldsymbol{\Theta}) = \frac{1}{2n} \sum_{k=1}^{n} |Y_k - \langle \boldsymbol{X}_k, \boldsymbol{\Theta} \rangle|^2 = \frac{1}{2} \operatorname{vec}(\boldsymbol{\Theta})^T \boldsymbol{\Sigma}_{XX} \operatorname{vec}(\boldsymbol{\Theta}) - \langle \boldsymbol{\Sigma}_{YX}, \boldsymbol{\Theta} \rangle + \operatorname{constant}$$

where $\Sigma_{XX} = \sum_{k=1}^{n} \operatorname{vec}(\boldsymbol{X}_k) \operatorname{vec}(\boldsymbol{X}_k)^T/n$, $\Sigma_{YX} = \sum_{k=1}^{n} Y_k \boldsymbol{X}_k/n$. However, this standard quadratic loss does not directly apply to our setting where full data are not available. In order to introduce some flexibility, we consider a generalized quadratic loss

$$\mathcal{L}(\boldsymbol{\Theta}) = \frac{1}{2} \operatorname{vec}(\boldsymbol{\Theta})^T \boldsymbol{Q} \operatorname{vec}(\boldsymbol{\Theta}) - \langle \boldsymbol{B}, \boldsymbol{\Theta} \rangle, \qquad (3.4)$$

where $\mathbf{Q} \in \mathbb{R}^{d^2 \times d^2}$ is symmetric, $\mathbf{B} \in \mathbb{R}^{d \times d}$. We present a framework for trace regression in Lemma 2. Note that [45, Theorem 1] is only for \mathbf{Q}, B in (3.4) being the (truncated) sample covariance, hence Lemma 2 can be viewed as its extension to more general \mathbf{Q}, B that suffices for our needs. Besides, our version is refined to be more technically amenable since a useful relation (3.6) is established even without the restricted strong convexity (3.7). One shall see that (3.6) can simplify the proofs of Theorems 9, 10, 12, 13.

Lemma 2. Consider trace regression (3.1) with (approximate) low-rankness (3.2), the estimator is given by (3.3) where the loss function is a generalized quadratic loss (3.4). Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. If Q is positive semi-definite, and λ satisfies

$$\lambda \ge 2 \| \max(\boldsymbol{Q} \cdot \operatorname{vec}(\boldsymbol{\Theta}^*)) - \boldsymbol{B} \|_{\operatorname{op}}, \tag{3.5}$$

then it holds that

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \le 10r^{\frac{1}{2-q}} \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{\frac{2-2q}{2-q}}.$$
(3.6)

Moreover, if the restricted strong convexity (RSC) holds, i.e., there exists $\kappa > 0$ such that

$$\operatorname{vec}(\widehat{\boldsymbol{\Delta}})^T \boldsymbol{Q} \operatorname{vec}(\widehat{\boldsymbol{\Delta}}) \ge \kappa \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^2,$$
(3.7)

then we have the convergence rate for Frobenius norm and nuclear norm

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}} \le 30\sqrt{r} \left(\frac{\lambda}{\kappa}\right)^{1-q/2} \text{ and } \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \le 300r \left(\frac{\lambda}{\kappa}\right)^{1-q}.$$
 (3.8)

With the preliminary of trace regression we now go into sparse linear regression

$$Y_k = X_k^T \Theta^* + \epsilon_k, \tag{3.9}$$

where $\Theta^* \in \mathbb{R}^d$ is the desired signal, X_k is the covariate (or sensing vector), ϵ_k is noise independent of X_k . In addition, Θ^* is approximately sparse.

Assumption 2. (Approximate sparsity on vector) For a specific $0 \le q < 1$, the desired signal $\Theta^* = [\theta_1^*, ..., \theta_d^*]^T$ satisfies

$$\sum_{i=1}^{d} |\theta_i^*|^q \le s.$$
 (3.10)

It is not hard to see that (3.9), (3.10) are encompassed by (3.1), (3.2) if X_k, Θ^* are diagonal, i.e., $X_k = \text{diag}(X_k), \Theta^* = \text{diag}(\Theta^*)$, so we consider analogue of (3.3) as the estimator. The first issue is the choice of loss function since the existing methods are invalid: we can neither use the quadratic loss as [45,70] without full data, nor the negative log-likelihood as [43] due to the noise ϵ_k with unknown distribution. Instead, we resort to a generalized quadratic loss given in (3.4) to proceed. For sparse linear regression, particularly, we let $\mathcal{L}(\Theta) = \frac{1}{2}\Theta^T Q\Theta - B^T\Theta$ where $Q \in \mathbb{R}^{d \times d}$ is symmetric, $B \in \mathbb{R}^d$. Thus, our estimator is given by

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \ \frac{1}{2} \Theta^T \boldsymbol{Q} \Theta - B^T \Theta + \lambda \|\Theta\|_1.$$
(3.11)

Lemma 2 implies the following Corollary.

Corollary 2. Consider linear regression (3.9) with (approximate) sparsity (3.10), the estimator $\widehat{\Theta}$ is given by (3.11). Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. If \boldsymbol{Q} is positive semi-definite, λ satisfies

$$\lambda \ge 2 \|\boldsymbol{Q}\boldsymbol{\Theta}^* - B\|_{\max},\tag{3.12}$$

then it holds that

$$\|\widehat{\Delta}\|_{1} \le 10s^{\frac{1}{2-q}} \|\widehat{\Delta}\|_{2}^{\frac{2-2q}{2-q}}.$$
(3.13)

Moreover, if for some $\kappa > 0$ we have the restricted strong convexity

$$\widehat{\Delta}^T \boldsymbol{Q} \widehat{\Delta} \ge \kappa \| \widehat{\Delta} \|_2^2, \tag{3.14}$$

then we have the error bound for ℓ_2 and ℓ_1 norm

$$\|\widehat{\Delta}\|_{2} \leq 30\sqrt{s} \left(\frac{\lambda}{\kappa}\right)^{1-q/2} \text{ and } \|\widehat{\Delta}\|_{1} \leq 300s \left(\frac{\lambda}{\kappa}\right)^{1-q}$$
(3.15)

It remains to properly specify Q, B in (3.11). Note that the expected quadratic risk is given by

$$\mathbb{E}|Y_k - X_k^T \Theta|^2 = \Theta^T \mathbb{E}(X_k X_k^T) \Theta - (\mathbb{E}(Y_k X_k))^T \Theta + \text{constant}$$

thus a general guideline to choose \boldsymbol{Q} , B is that \boldsymbol{Q} should be close to the covariance matrix of X_k , and B should well approximate the covariance $\mathbb{E}(Y_k X_k)$. Naturally, based on 1-bit data we can still use $\hat{\boldsymbol{\Sigma}}$ in (2.4) or $\boldsymbol{\check{\Sigma}}$ in (2.1) as \boldsymbol{Q} . Nevertheless, the issue is that they may not be positive semi-definite, while the positive semi-definiteness of \boldsymbol{Q} is an indispensable condition in Corollary 2. To resolve the issue, we assume $\boldsymbol{\Sigma}_{XX} = \mathbb{E}X_k X_k^T$ is column-wisely sparse. We defer an illustration of this assumption to Remark 2.

Assumption 3. $X_1, ..., X_n$ are *i.i.d.* drawn from a zero-mean random vector with covariance matrix $\Sigma_{XX} = \mathbb{E}X_k X_k^T = [\sigma_{ij}]$ satisfying Assumption 1 under parameter $(0, s_0)$, *i.e.*, the number of non-zero elements in each column is less than s_0 . Besides, Σ_{XX} is positive definite, and for some absolute constant $\kappa_0 > 0$ it satisfies $\lambda_{\min}(\Sigma_{XX}) \ge 2\kappa_0$.

Under Assumption 3, our estimator $\widehat{\Sigma}$ defined in (2.4) for sub-Gaussian data, or (2.18) for heavy-tailed data, is positive definite with high probability. Thus, we set $\mathbf{Q} = \widehat{\Sigma}$ in (3.11). Note that $\mathbb{E}(Y_k X_k)$ is also covariance, enlightened by Corollary 1, we similarly set

$$\widehat{\Sigma}_{YX} = \frac{1}{n} \sum_{k=1}^{n} \gamma^2 \cdot \dot{Y}_k \dot{X}_{k1}.$$
(3.16)

Now we have specified our estimator as

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \quad \frac{1}{2} \Theta^T \widehat{\Sigma} \Theta - \widehat{\Sigma}_{YX}^T \Theta + \lambda \|\Theta\|_1.$$
(3.17)

3.1 Sub-Gaussian Data

We assume the sub-Gaussian, zero-mean covariate and sub-Gaussian noise satisfying $||X_k||_{\psi_2} \leq \sigma_1$, $||\epsilon_k||_{\psi_2} \leq \sigma_2$, and $||\Theta^*||_2 \leq R = O(1)$. In this setting, we have $||Y_k||_{\psi_2} \leq ||X_k^T\Theta^*||_{\psi_2} + ||\epsilon_k||_{\psi_2} \leq ||\Theta^*||_2 ||X_k||_{\psi_2} + ||\epsilon_k||_{\psi_2} = O(\max\{\sigma_1, \sigma_2\})$. To lighten notations without losing generality, we assume for some $\sigma > 0$

$$\max\left\{\|X_k\|_{\psi_2}, \|Y_k\|_{\psi_2}\right\} \le \sigma \tag{3.18}$$

and use the uniform noise with the same dithering scale γ to dither X_k and Y_k before 1bit quantization. More precisely, we choose dithering noise $\Gamma_{k1}, \Gamma_{k2} \sim \text{uni}([-\gamma, \gamma]^d), \Lambda_k \sim \text{uni}([-\gamma, \gamma])$ with γ in (2.6), then we obtain the 1-bit data $(\dot{X}_{k1}, \dot{X}_{k2}, \dot{Y}_k)$.

We mention that our result directly extends to more general setting where $||X_k||_{\psi_2}$, $||Y_k||_{\psi_2}$ may vary a lot. Indeed, we can adaptively choose dithering scale according to $||X_k||_{\psi_2}$ and $||Y_k||_{\psi_2}$, for instance, $\Gamma_{k1}, \Gamma_{k2} \sim \operatorname{uni}([-\gamma_X, \gamma_X]^d), \Lambda_k \sim \operatorname{uni}([\gamma_Y, \gamma_Y])$. In our numerical simulations, we also applied different dithering scales to X_k and Y_k to improve the recovery.

In Theorem 7 we will give the near minimax statistical rate for the estimator Θ . The idea is to invoke Corollary 2, and this requires (3.12) and (3.14). To properly set λ to confirm (3.12), it suffices to bound $\|\widehat{\Sigma}\Theta^* - \widehat{\Sigma}_{YX}\|_{\text{max}}$ from above. Combining Assumption 3 and results in Section 2, we can show (3.14) holds with high probability.

Theorem 7. Assume (3.9), Assumption 2, (3.18) hold, $\|\Theta^*\|_2 \leq R$ for some absolute constant R, and Assumption 3 holds for some fixed integer s_0 . Before the quantization we dither the data with γ in (2.6). We consider $\widehat{\Theta}$ given by (3.17) where $\widehat{\Sigma}, \widehat{\Sigma}_{YX}$ are respectively set as (2.4), (3.16), and ζ is given by (2.9). Moreover, we choose λ by

$$\lambda = C_6 \log n \sqrt{\frac{\delta \log d}{n}} \tag{3.19}$$

with sufficiently large C_6 . Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. When $\frac{(\log n)^2 \log d}{n}$ is sufficiently small, with probability at least $1 - \exp(-\delta) - 2d^{2-\delta}$, we have

$$\begin{cases} \|\widehat{\Delta}\|_{2} \lesssim \sqrt{s} \left(\sigma^{2} \log n \sqrt{\frac{\delta \log d}{n}}\right)^{1-q/2} \\ \|\widehat{\Delta}\|_{1} \lesssim s \left(\sigma^{2} \log n \sqrt{\frac{\delta \log d}{n}}\right)^{1-q} \end{cases}$$
(3.20)

Remark 2. Compared to the sample covariance $\sum_{k=1}^{n} X_k^T X_k/n$, the proposed 1-bit covariance matrix estimator $\widehat{\Sigma}$ lacks positive semi-definiteness. We address the issue by assuming columnwise sparsity of Σ_{XX} , which together with $\lambda_{\min}(\Sigma_{XX}) = \Omega(1)$ can provide positive definiteness under high-dimensional scaling. This assumption is also used in [98] to resolve the same issue. As an example, this accommodates isotropic sensing vectors that is conventionally adopted in compressed sensing literature [25, 40, 77]. In addition, we have removed this (a bit uncommon) assumption in our subsequent work [30].

3.2 Heavy-tailed Data

We then switch to the heavy-tailed case where X_k and ϵ_k are only assumed to possess bounded 4-th moment. We consider the scaling of the desired signal as $\|\Theta^*\|_2 \leq R = O(1)$. Moreover, we assume $\mathbb{E}|V^T X_k|^4 \leq M_1$ for any $V \in \mathbb{R}^d$, $\|V\|_2 \leq 1$, and $\mathbb{E}|\epsilon_k|^4 \leq M_2$. Then we have the fourth moment of Y_k is also bounded by $O(R^4M_1 + M_2)$. To lighten the notation without losing generality, we assume the same upper bound M for covariate and response:

$$\max\left\{\sup_{\|V\|_2 \le 1} \mathbb{E}|V^T X_k|^4, \mathbb{E}|Y_k|^4\right\} \le M,\tag{3.21}$$

which allows us to use the same truncation parameter η and dithering scale γ for X_k and Y_k .

Similar to the same comment for sub-Gaussian case, if the fourth moment of X_k and Y_k have different scales, our method still works under different truncation parameters (η_X, η_Y) and dithering parameters (γ_X, γ_Y) . Moreover, it is straightforward to adapt our method to the mixing case studied in [45] where X_k is sub-Gaussian but ϵ_k (and hence Y_k) is heavy-tailed. In this mixing setting, only the responses are treated as heavy-tailed data and truncated before dithering.

Theorem 8. Assume (3.9), Assumption 2, (3.21) hold, $\|\Theta^*\|_2 \leq R$ for some absolute constant R, and Assumption 3 holds for some fixed integer s_0 . By setting η, γ as (2.15) such that $\gamma > \eta$, we first truncate (X_k, Y_k) element-wisely to $(\widetilde{X}_k, \widetilde{Y}_k)$ with parameter η , then dither the truncated data with uniform noise on $[-\gamma, \gamma]$ and quantize the data to $(\dot{X}_{k1}, \dot{X}_{k2}, \dot{Y}_k)$ finally. We consider $\widehat{\Theta}$ in (3.17) where $\widehat{\Sigma}$ is given in (2.18) with ζ set as (2.19), $\widehat{\Sigma}_{YX}$ is given in (3.16). Moreover, we choose

$$\lambda = C_7 \sqrt{M} \left(\frac{\delta \log d}{n}\right)^{1/4} \tag{3.22}$$

with sufficiently large C_7 . Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. When $\log d/n$ is sufficiently small, with probability at least $1 - \exp(-\delta) - 2d^{2-d}$, we have

$$\begin{cases} \|\widehat{\Delta}\|_{2} \lesssim \sqrt{s} M^{1/2-q/4} \left(\frac{\delta \log d}{n}\right)^{1/4-q/8} \\ \|\widehat{\Delta}\|_{1} \lesssim s M^{(1-q)/2} \left(\frac{\delta \log d}{n}\right)^{(1-q)/4} \end{cases}$$
(3.23)

We emphasize that our method does not rely on the full knowledge of Σ_{XX} ; indeed, our method applies as long as Σ_{XX} satisfies Assumption 3. Note that when Σ_{XX} is known as a priori, we can directly set $Q = \Sigma_{XX}$ in (3.11), and the same error rates can be obtained by the similar techniques.

3.3 1-bit Compressed Sensing

We just studied sparse linear regression based on the 1-bit quantized covariates and responses $(\dot{X}_{k1}, \dot{X}_{k2}, \dot{Y}_k)$, while the only related problem studied in existing works is 1-bit compressed sensing (1-bit CS). In 1-bit CS, one considers the same linear model (3.9) and wants to estimate the sparse underlying signal Θ^* based on (X_k, \dot{Y}_k) , where X_k denotes the full covariate, and $\dot{Y}_k \in \{-1, 1\}$ is the 1-bit quantized version of the response Y_k . In particular, earlier works mainly studied a direct quantization with $\dot{Y}_k = \text{sign}(X^T \Theta^*)$ (see, e.g., [15, 55, 74, 75]), while

recent works (e.g., [5, 39, 40, 63, 89]) began to consider dithered quantization that are more relevant to our work, i.e., $\dot{Y}_k = \text{sign}(X^T\Theta^* + \Lambda)$ for some dithering noise Λ . Specifically, by the additional dithering step, these works overcome several limitations and present better results. For instance, full reconstruction with norm [63], exponentially-decaying error rate [5], and extension to non-Gaussian sensing vectors [39, 40, 89].

Since one still has full knowledge on X_k in 1-bit CS, our problem setting is novel and evidently more tricky. From a practical viewpoint, due to the binary covariate, the storage and communication costs are further lowered in our method. Technically, the key element that allows quantization of covariate is the new 1-bit sparse covariance matrix estimator developed in Section 2. To facilitate presentation and future study, we term this new setting as 1-bit quantized-covariate compressed sensing (1-bit QC-CS) to distinguish with the canonical 1-bit CS.

Note that it is unfair to compare our Theorem 7, 8 with existing results for 1-bit CS. To see the contributions of this paper more explicitly, we analogously establish results for 1-bit CS where full-precision X_k are available, under both sub-Gaussian and heavy-tailed regimes. Similar to (3.17), we formulate the estimation as a convex programming problem, but substitute $\hat{\Sigma}$ in (3.17) with the sample covariance matrix $\hat{\Sigma}_{XX} = \sum_{k=1}^{n} X_k X_k^T / n$ for sub-Gaussian X_k , or the truncated sample covariance matrix $\hat{\Sigma}_{\tilde{X}\tilde{X}} = \sum_{k=1}^{n} \tilde{X}_k \tilde{X}_k^T / n$ for heavytailed X_k . Here, for heavy-tailed case, we truncate X_k element-wisely, but we distinguish the truncation threshold of X_k , Y_k by different notations η_X , η_Y and they are set to be different values. More precisely, the *i*-th entry of \tilde{X}_k is given by $\tilde{X}_{k,i} = \text{sign}(X_{k,i}) \min\{|X_{k,i}|, \eta_X\}$, while before the dithered quantization, Y_k is truncated to be $\tilde{Y}_k = \text{sign}(Y_k) \min\{|Y_k|, \eta_Y\}$.

Although the results are similarly established by the framework of trace regression, we feel obliged to note some differences. Let us consider the sub-Gaussian regime for illustration. Firstly, the column-wise sparsity of Σ_{XX} in Assumption 3, whose main aim is to guarantee positive semi-definiteness of the 1-bit covariance matrix estimator $\hat{\Sigma}$, can now be removed as $\hat{\Sigma}_{XX}$ is automatically positive semi-definite. But on the other hand, without this assumption, we no longer have a dimension-free upper bound on $\|\hat{\Sigma}_{XX} - \Sigma_{XX}\|_{op}$, hence the proof cannot proceed to (B.11). Indeed, we only have dimension-free upper bound on $\|\hat{\Sigma}_{XX} - \Sigma_{XX}\|_{max}$. In heavy-tailed case we hence impose a stronger scaling $\|\Theta^*\|_1 \leq R$, (which is also used in the heavy-tailed case of sparse linear regression in [45, Lemma 1(b)]). In addition, we need to establish the restricted strong convexity (3.7) in Lemma 2 via some additional technicalities.

In the next two theorems we present our results on 1-bit CS, which are directly comparable to the prior results of 1-bit CS. To facilitate the flow of our presentation, a detailed comparison is postponed to Appendix D. One shall see that, the following two results improve on existing ones from some respect.

Theorem 9. (1-bit CS with sub-Gaussian data). Assume (3.9), Assumption 2 hold, $||X_k||_{\psi_2} \leq \sigma_1$, $||\epsilon_k||_{\psi_2} \leq \sigma_2$, $||\Theta^*||_2 \leq R$ with absolute constants σ_1 , σ_2 , R. For the zero-mean covariate X_k , define the covariance matrix $\Sigma_{XX} = \mathbb{E}X_k X_k^T$ and we assume $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$ for some absolute constant $\kappa_0 > 0$. We quantize Y_k to be $\dot{Y}_k = \text{sign}(Y_k + \Lambda_k)$ with Λ_k uniformly distributed on $[-\gamma, \gamma]$, and we set $\gamma = C'_8 \sqrt{\log n}$ for sufficiently large C'_8 . The estimation is formulated as a convex programming problem

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \quad \frac{1}{2} \Theta^T \widehat{\Sigma}_{XX} \Theta - \widehat{\Sigma}_{YX}^T \Theta + \lambda \|\Theta\|_1.$$
(3.24)

Moreover, we set $\widehat{\Sigma}_{XX}$, $\widehat{\Sigma}_{YX}^{T}$ and λ in (3.24) as

$$\widehat{\Sigma}_{XX} = \frac{1}{n} \sum_{k=1}^{n} X_k X_k^T, \quad \widehat{\Sigma}_{YX} = \frac{1}{n} \sum_{k=1}^{n} \gamma \cdot \dot{Y}_K X_k, \quad \lambda = C_8 \sqrt{\frac{\delta \log d \log n}{n}}$$
(3.25)

with some sufficiently large C_8 . Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$, then when $s\left(\frac{\delta \log d}{n}\right)^{1-q/2}$ is sufficiently small, with probability at least $1 - 7d^{2-\delta}$, we have

$$\begin{cases} \|\widehat{\Delta}\|_{2} \lesssim \sqrt{s} \left(\sqrt{\frac{\delta \log d \log n}{n}}\right)^{1-q/2} \\ \|\widehat{\Delta}\|_{1} \lesssim s \left(\sqrt{\frac{\delta \log d \log n}{n}}\right)^{1-q} \end{cases}$$
(3.26)

By taking advantage of the full covariate, in heavy-tailed regime of 1-bit CS we show the ℓ_2 norm error rate $O\left(s^{2/3}\left(\frac{\log d}{n}\right)^{1/3-q/6}\right)$, which is faster than the corresponding rate for 1-bit QC-CS in Theorem 8.

Theorem 10. (1-bit CS with heavy-tailed data). Assume (3.9), Assumption 2, (3.21) hold (M in (3.21) is an absolute constant), $\|\Theta^*\|_1 \leq R$ for some absolute constant R. For the zeromean covariate X_k we let $\Sigma_{XX} = \mathbb{E}X_k X_k^T$ and assume $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$ for some absolute constant $\kappa_0 > 0$. We element-wisely truncate X_k to be \widetilde{X}_k with threshold η_X , and truncate Y_k to be \widetilde{Y}_k with threshold η_Y . Then, \widetilde{Y}_k is dithered and quantized to be $\dot{Y}_k = \operatorname{sign}(\widetilde{Y}_k + \Lambda_k)$ with Λ_k uniformly distributed on $[-\gamma, \gamma]$. For specific $\delta > 0$, we set these signal processing parameters as

$$\eta_X = C_9 \left(\frac{n}{\delta \log d}\right)^{\frac{1}{4}}, \quad \eta_Y = C_{10} \left(\frac{n}{\delta \log d}\right)^{\frac{1}{6}}, \quad \gamma = C_{11} \left(\frac{n}{\delta \log d}\right)^{\frac{1}{6}}, \tag{3.27}$$

where $C_{11} > C_{10}$ to give $\gamma > \eta_Y$. The estimation is formulated as a convex programming problem

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \ \frac{1}{2} \Theta^T \widehat{\Sigma}_{\tilde{X}\tilde{X}} \Theta - \widehat{\Sigma}_{YX}^T \Theta + \lambda \|\Theta\|_1.$$
(3.28)

Moreover, we set $\widehat{\Sigma}_{\tilde{X}\tilde{X}}$, $\widehat{\Sigma}_{YX}$ and λ in (3.28) as

$$\widehat{\Sigma}_{\tilde{X}\tilde{X}} = \frac{1}{n} \sum_{k=1}^{n} \widetilde{X}_k \widetilde{X}_k^T, \quad \widehat{\Sigma}_{YX} = \frac{1}{n} \sum_{k=1}^{n} \gamma \cdot \dot{Y}_k \widetilde{X}_k, \quad \lambda = C_{12} \left(\frac{\delta \log d}{n}\right)^{1/3}$$
(3.29)

with some sufficiently large C_{12} . Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. Under sufficiently small $s^2 \left(\frac{\delta \log d}{n}\right)^{1-q/2}$, and we further assume $s \left(\frac{\delta \log d}{n}\right)^{\frac{1}{2}-\frac{q}{3}} = O(1)$ for $q \in (0, 1)$, then with probability at least $1 - O(d^{2-\sqrt{\delta}})$, we have

$$\begin{cases} \|\widehat{\Delta}\|_2 \lesssim \sqrt{s} \left(\frac{\delta \log d}{n}\right)^{\frac{1-q/2}{3}} \\ \|\widehat{\Delta}\|_1 \lesssim s \left(\frac{\delta \log d}{n}\right)^{\frac{1-q}{3}} \end{cases}$$
(3.30)

We conclude this section by deriving an information-theoretic lower bound regarding Theorem 10. To be concise we only deal with exactly sparse Θ^* , i.e., *s*-sparse Θ^* . As Theorem 10 we consider the set of parameters

$$\mathscr{K}(s,R) = \left\{ \Theta \in \mathbb{R}^d : \|\Theta\|_0 \le s, \|\Theta\|_1 \le R \right\}.$$
(3.31)

Theorem 11. Given n, d, s, R and covariates $\{X_k : k \in [n]\}$, and assume $s \leq \frac{d}{8}$. For some underlying $\Theta \in \mathscr{K}(s, R)$, we suppose the responses $(\dot{Y}_k)_{k=1}^n$ are obtained as in Theorem 10, i.e., truncation, dithering and quantization with parameters in (3.29), and we assume $\eta_Y < \frac{8}{9}\gamma$. Consider any algorithm which, for any underlying $\Theta \in \mathscr{K}(s, R)$, takes $\{X_k : k \in [n]\}$ and the corresponding $(\dot{Y}_k)_{k=1}^n$ as input and returns $\widehat{\Theta}$. If $n \gtrsim K_u^{-1}(\frac{s}{R})^3 \log \frac{d}{2s}$, and $\sum_{k=1}^n |X_k^T V|^2 \leq nK_u ||V||_2^2$ holds for some K_u and for all 2s-sparse $V \in \mathbb{R}^d$, then there exists $\Theta_0 \in \mathscr{K}(s, R)$ such that with probability at least $\frac{3}{4}$,

$$\|\widehat{\Theta} - \Theta_0\|_2 \ge \left(\frac{\log \frac{d}{2s}}{\log d}\right)^{\frac{1}{6}} \sqrt{\frac{s}{K_u}} \left(\frac{\log \frac{d}{2s}}{n}\right)^{1/3}.$$
(3.32)

The proof follows similar courses as in [80, Theorem 1], while the main difference is on bounding the Kullback-Leibler divergence because of the 1-bit observation \dot{Y}_k . The conditions are quite benign: $\sum_{k=1}^{n} |X_k^T V|^2 \leq nK_u ||V||_2^2$ for 2s-sparse V is satisfied by a large class of random X_k with bounded or logarithmic K_u (e.g., i.i.d., sub-Gaussian or sub-exponential X_k); $n \geq K_u^{-1} \left(\frac{s}{R}\right)^3 \log \frac{d}{2s}$ is in the interesting high-dimensional scaling and assumed to guarantee the packing set is a subset of $\mathscr{K}(s, R)$. Note that the lower bound matches our upper bound for ℓ_2 error in Theorem 10, up to logarithmic factors and the parameter K_u . Thus, for estimation of Θ^* from the observed data (X_k, \dot{Y}_k) , the rate in Theorem 10 is almost tight, and significantly faster rate is not achievable without changing the process of producing \dot{Y}_k .

In the proof, the key point that lifts the regular lower bound $\tilde{\Omega}(\frac{\sqrt{s}}{\sqrt{n}})$ to $\tilde{\Omega}(\frac{\sqrt{s}}{n^{1/3}})$ is our choice of dithering scale (i.e., $\gamma \asymp \left(\frac{n}{\log d}\right)^{1/6}$). Note that it is essentially larger than $\gamma \asymp \sqrt{\log n}$ used in the sub-Gaussian case (Theorem 9). Such a larger dithering scale can be understood as the price we pay for dealing with heavy-tailed noise.

4 Low-rank Matrix Completion

Matrix completion refers to the problem of recovering a low-rank matrix with incomplete observations of the entries, which is motivated by recommendation system, system identification, quantum state tomography, image inpainting, and many others, see [7,28,31,32,35,46,52] for instance. The literature can be roughly divided into two lines, exact recovery and approximate recovery (i.e., statistical estimation). To establish exact recovery guarantee, the underlying matrix is required to satisfy a quite stringent incoherence condition proposed and developed in [24, 26, 27, 81]. By contrast, it was shown that matrix with low spikiness could be well approximated (or estimated) under much more relaxed condition [60, 64, 71]. This Section is intended to study the estimation problem of matrix completion via the binary data produced by our 1-bit quantization scheme. For simplicity we consider square matrix and formulate the model as

$$Y_k = \langle \boldsymbol{X_k}, \boldsymbol{\Theta^*} \rangle + \epsilon_k \tag{4.1}$$

where Θ^* is the underlying low-rank data matrix of interest, X_k distributed on $\{e_i e_j^T : i, j \in [d]\}$ is the sampler that extracts one entry of Θ^* , Y_k is the k-th observation corrupted by noise ϵ_k independent of X_k . We consider a random, uniform sampling scheme

$$X_1, ..., X_n$$
 are i.i.d. uniformly distributed on $\{e_i e_j^T : i \in [d], j \in [d]\},$ (4.2)

but we mention that the results can be directly adapted to non-uniform sampling scheme, see [60]. To embrace more real applications, Θ^* is assumed to be approximately low-rank [71].

Assumption 4. (Approximate low-rankness on matrix). Let $\sigma_1(\Theta^*) \ge \dots \ge \sigma_d(\Theta^*)$ be singular values of Θ^* , $0 \le q < 1$. For some r > 0 it holds that

$$\sum_{k=1}^{d} \sigma_k(\mathbf{\Theta}^*)^q \le r.$$
(4.3)

Since X_k only has d^2 values, we can use $\lceil 2 \log_2 d \rceil$ bits to encode X_k without losing any information. Therefore, we only quantize Y_k to binary data \dot{Y}_k and study the estimation via (X_k, \dot{Y}_k) . Similar to our prior developments, we use a generalized quadratic loss (3.4) with Q, B specified to be

$$\boldsymbol{Q} = \frac{1}{n} \sum_{k=1}^{n} \operatorname{vec}(\boldsymbol{X}_{\boldsymbol{k}}) \operatorname{vec}(\boldsymbol{X}_{\boldsymbol{k}})^{T} , \quad \boldsymbol{B} = \frac{1}{n} \sum_{k=1}^{n} \gamma \cdot \dot{Y}_{k} \boldsymbol{X}_{\boldsymbol{k}}.$$
(4.4)

The spikiness of Θ^* is defined as $\frac{d\|\Theta^*\|_{\max}}{\|\Theta^*\|_F}$ in [71], and note that completing a matrix with high spikiness (close to d) is an ill-posed problem per se [35]. Besides the spikiness, a similar but more straightforward assumption is a max-norm constraint (e.g., [23,28,34,60]). Here, we adopt this more straightforward condition and assume

$$\|\mathbf{\Theta}^*\|_{\max} \le \alpha^*. \tag{4.5}$$

Substitute (4.4), (3.4) into (3.3), together with the max-norm constraint (4.5), we now define our estimator via the following convex programming problem

$$\widehat{\boldsymbol{\Theta}} \in \underset{\|\boldsymbol{\Theta}\|_{\max} \leq \alpha^{*}}{\operatorname{arg\,min}} \quad \frac{1}{2} \operatorname{vec}(\boldsymbol{\Theta})^{T} \boldsymbol{Q} \operatorname{vec}(\boldsymbol{\Theta}) - \langle \boldsymbol{B}, \boldsymbol{\Theta} \rangle + \lambda \|\boldsymbol{\Theta}\|_{\operatorname{nu}}$$

$$= \underset{\|\boldsymbol{\Theta}\|_{\max} \leq \alpha^{*}}{\operatorname{arg\,min}} \quad \frac{1}{2n} \sum_{k=1}^{n} \left(\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta} \rangle - \gamma \cdot \dot{Y}_{k} \right)^{2} + \lambda \|\boldsymbol{\Theta}\|_{\operatorname{nu}}$$

$$(4.6)$$

Compared with the program (3.17) involving $\widehat{\Sigma}$ used in sparse linear regression, (4.6) is more intuitive since we simply replace the full observation Y_k in a standard quadratic loss with its 1-bit surrogate $\gamma \cdot \dot{Y}_k$. Such choice can be readily explained by Lemma 1.

Applying Lemma 2 to the problem set-up of low-rank matrix completion directly gives the following Corollary 3.

Corollary 3. Consider (4.1) under random sampling (4.2), Θ^* satisfies Assumption 4 and (4.5). Consider $\widehat{\Theta}$ in (4.6). Let $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. If

$$\lambda \ge 2 \left\| \frac{1}{n} \sum_{k=1}^{n} \left[\left\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta}^* \right\rangle - \gamma \cdot \dot{Y}_k \right] \boldsymbol{X}_{\boldsymbol{k}} \right\|_{\text{op}},$$
(4.7)

then it holds that

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \le 10r^{\frac{1}{2-q}} \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{\frac{2-2q}{2-q}}.$$
(4.8)

Moreover, if the RSC holds, i.e., for some $\kappa > 0$

$$\frac{1}{n}\sum_{k=1}^{n}|\langle \boldsymbol{X}_{\boldsymbol{k}}, \widehat{\boldsymbol{\Delta}} \rangle|^{2} \ge \kappa \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{2}, \tag{4.9}$$

then we have

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}} \le 30\sqrt{r} \left(\frac{\lambda}{\kappa}\right)^{1-q/2} \text{ and } \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \le 300r \left(\frac{\lambda}{\kappa}\right)^{1-q}.$$
 (4.10)

4.1 Sub-Gaussian noise

We first consider sub-Gaussian noise ϵ_k satisfying

$$\mathbb{E}\epsilon_k = 0, \ \|\epsilon_k\|_{\psi_2} \le \sigma. \tag{4.11}$$

To invoke Corollary 3 and obtain the statistical rate, we need to choose suitable λ that guarantees (4.7) with high probability. Thus, we upper bound the right hand side of (4.7) first.

Lemma 3. Consider (4.1) under sampling scheme (4.2), max-norm constraint (4.5), and sub-Gaussian noise assumption (4.11). For a specific $\delta > 1$, we choose the dithering noise scale γ by

$$\gamma = C_{13} \max\{\alpha^*, \sigma\} \sqrt{\log\left(\frac{n}{\delta d \log(2d)}\right)}$$
(4.12)

with some sufficiently large C_{13} such that $\gamma \geq 2 \max\{\alpha^*, \sigma\}$. If $\frac{\delta d \log d}{n}$ is sufficiently small, we have

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle-\gamma\cdot\dot{Y}_{\boldsymbol{k}}\right]\boldsymbol{X}_{\boldsymbol{k}}\right\|_{\mathrm{op}}\lesssim\max\{\alpha^{*},\sigma\}\sqrt{\log n\frac{\delta\log d}{nd}}$$
(4.13)

with probability higher than $1 - 2d^{1-\delta}$.

It remains to consider (4.9). To lighten the notation we use $\mathscr{X} = (X_1, ..., X_n)$ to denote the observed positions and define $\mathcal{F}_{\mathscr{X}}(\Theta) = n^{-1} \sum_{k=1}^n |\langle X_k, \Theta \rangle|^2$. It is known that $\mathcal{F}_{\mathscr{X}}(\Theta) \ge \kappa \|\widehat{\Delta}\|_F^2$ may not always hold under high-dimensional scaling and the special covariate (4.2). In this case, one often needs to establish (4.9) with a relaxed (tolerance) term [72, Definition 2]. To this end, Negahban and Wainwright first established such relaxed RSC over a constraint set in [71, Theorem 1]. Later, in [60, Lemma 12], Klopp considered a different constraint set and provided a refined proof, but only for the exact low-rank setting, i.e., q = 0 in Assumption 4. More recently, in [28, Lemma 5], Chen and Ng considered a constraint set depending on $q \in [0, 1)$ and extended the proof in [60] to approximate low-rank regime. As a consequence, a simpler and much shorter proof for the error bound in [71] could be obtained, see more discussions in [28]. Here we show the relaxed RSC over the constraint set defined in [28], see $\mathcal{C}(\psi)$ in (4.14). **Lemma 4.** For a specific δ and sufficiently large ψ , we consider the constraint set

$$\mathcal{C}(\psi) = \left\{ \boldsymbol{\Theta} \in \mathbb{R}^{d \times d} : \|\boldsymbol{\Theta}\|_{\max} \le 2\alpha^*, \|\boldsymbol{\Theta}\|_{\mathrm{nu}} \le 10r^{\frac{1}{2-q}} \|\boldsymbol{\Theta}\|_{\mathrm{F}}^{\frac{2-2q}{2-q}}, \\ \|\boldsymbol{\Theta}\|_{\mathrm{F}}^2 \ge (\alpha^* d)^2 \sqrt{\frac{\psi\delta \log(2d)}{n}} \right\}.$$

$$(4.14)$$

Then there exists some absolute constant $\kappa \in (0, 1)$, such that with probability at least $1 - d^{-\delta}$, it holds that

$$\mathcal{F}_{\mathscr{X}}(\mathbf{\Theta}) \ge \kappa d^{-2} \|\mathbf{\Theta}\|_{\mathrm{F}}^{2} - T_{0}, \ \forall \mathbf{\Theta} \in \mathcal{C}(\psi),$$
(4.15)

where the relaxation term T_0 is given by

$$T_0 = \frac{r}{(2-q)d^q} \left(240\alpha^* \sqrt{\frac{d\log(2d)}{n}} \right)^{2-q}.$$
 (4.16)

We are now ready to derive the statistical bound of the estimation error $\widehat{\Delta} = \widehat{\Theta} - \Theta^*$. The main idea is parallel to previous works [28,60,71], i.e., to discuss whether $\widehat{\Delta}$ belongs to $\mathcal{C}(\psi)$. Note that this only hinges on the third constraint in (4.14), since the first two constraints are automatically satisfied by $\widehat{\Delta}$, see (4.5) and (4.8).

Theorem 12. Under the setting of Lemma 3, assume Θ^* satisfies Assumption 4, we consider the estimator $\widehat{\Theta}$ defined in (4.6). Moreover, we set λ by

$$\lambda = C_{14} \max\{\alpha^*, \sigma\} \sqrt{\log n \frac{\delta \log d}{nd}}$$
(4.17)

with sufficiently large C_{14} , assume $\frac{\delta d \log d}{n}$ is sufficiently small, $r \gtrsim d^q$, $n \lesssim d^2 \log(2d)$, then with probability higher than $1 - 3d^{1-\delta}$, we have

$$\begin{cases} \|\widehat{\mathbf{\Delta}}\|_{\mathrm{F}}^{2}/d^{2} \lesssim rd^{-q} \Big(\max\{(\alpha^{*})^{2}, \sigma^{2}\} \log n \frac{\delta d \log d}{n} \Big)^{1-q/2} \\ \|\widehat{\mathbf{\Delta}}\|_{\mathrm{nu}}/d \lesssim rd^{-q} \Big(\max\{\alpha^{*}, \sigma\} \sqrt{\log n \frac{\delta d \log d}{n}} \Big)^{1-q}. \end{cases}$$

$$\tag{4.18}$$

Remark 3. Under a specific scaling $\|\Theta^*\|_{\rm F} = 1$, $X_k = d \cdot e_i e_j^T$ adopted in [45, 71], our bound for the mean square error $d^{-2} \|\widehat{\Delta}\|_{\rm F}^2$ is equivalent to

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{2} \lesssim r \Big(\max\{\alpha(\boldsymbol{\Theta}^{*})^{2}, \sigma^{2}\} \frac{d \log d \log n}{n} \Big)^{1-q/2},$$

where $\alpha(\Theta^*) = \frac{d\|\Theta^*\|_{\max}}{\|\Theta^*\|_F} \in [1, d]$ is the spikiness of the desired Θ^* . Compared with the full-databased estimator in [71] that achieves near minimax rate [71, Theorem 3], our 1-bit estimator only degrades by a minor factor $\log n$, hence is also near minimax. It is quite striking that the underlying matrix can be recovered fairly well from merely 1-bit observation.

4.2 Heavy-tailed noise

The heavy-tailed noise is assumed to have bounded second moment in this part, i.e.,

$$\mathbb{E}\epsilon_k = 0, \ \mathbb{E}|\epsilon_k|^2 \le M. \tag{4.19}$$

Note that the 1-bit response \dot{Y}_k is obtained with the truncation step before the dithered quantization — Y_k is first truncated to be $\tilde{Y}_k = \operatorname{sign}(Y_k) \min\{|Y_k|, \eta\}$, then dithered and quantized to $\dot{Y}_k = \operatorname{sign}(\tilde{Y}_k + \Lambda_k)$ where $\Lambda_k \sim \operatorname{uni}([-\gamma, \gamma])$. To invoke Corollary 3, we first upper bound the right hand side of (4.7).

Lemma 5. Consider (4.1) under sampling scheme (4.2), max-norm constraint (4.5), and heavy-tailed noise assumption (4.19). For a specific $\delta > 1$, we set the truncation threshold η , dithering scale γ as

$$\begin{cases} \eta = C_{15} \max\{\alpha^*, \sqrt{M}\} \left(\frac{n}{\delta d \log d}\right)^{1/4} \\ \gamma = C_{16} \max\{\alpha^*, \sqrt{M}\} \left(\frac{n}{\delta d \log d}\right)^{1/4} \end{cases}, \tag{4.20}$$

where $C_{16} > C_{15}$, $\gamma > 2 \max\{\alpha^*, \sqrt{M}\}$. If $\frac{\delta d \log d}{n}$ is sufficiently small, we have

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle-\gamma\cdot\dot{Y}_{\boldsymbol{k}}\right]\boldsymbol{X}_{\boldsymbol{k}}\right\|_{\mathrm{op}}\lesssim\max\{\alpha^{*},\sqrt{M}\}\left(\frac{\delta\log d}{nd^{3}}\right)^{1/4}$$
(4.21)

with probability higher than $1 - 2d^{1-\delta}$.

Parallel to proof of Theorem 12, a discussion on whether $\widehat{\Delta} \in \mathcal{C}(\psi)$ unfolds some key relations that further lead to the desired error bounds. The result is given in Theorem 13.

Theorem 13. Under the setting of Lemma 5, assume Θ^* satisfies Assumption 4, we consider the estimator $\widehat{\Theta}$ defined in (4.6). Moreover, we set λ as

$$\lambda = C_{17} \max\{\alpha^*, \sqrt{M}\} \left(\frac{\delta \log d}{nd^3}\right)^{1/4}$$
(4.22)

with sufficiently large C_{17} . Assume $\frac{\delta d \log d}{n}$ is sufficiently small, $r \gtrsim d^q$, then with probability at least $1 - 3d^{1-\delta}$, we have

$$\begin{cases} \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{2}/d^{2} \lesssim rd^{-q} \Big(\max\{(\alpha^{*})^{2}, M\} \sqrt{\frac{\delta d \log d}{n}} \Big)^{1-q/2} \\ \|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}}/d \lesssim rd^{-q} \Big(\max\{\alpha^{*}, \sqrt{M}\} \Big(\frac{\delta d \log d}{n} \Big)^{1/4} \Big)^{1-q} \end{cases}$$
(4.23)

The result is consistent with previous two estimation problems — the error rates become essentially slower in the 1-bit heavy-tailed case. Similarly, this is the outcome of a bias-andvariance trade-off.

To shed some light on the fundamental difficulty of estimating Θ^* from (X_k, \dot{Y}_k) , we derive an information-theoretic lower bound in the following. Specifically, we consider the exact lowrank case (q = 0) with the set of parameters

$$\mathscr{K}(r,\alpha^*) = \left\{ \boldsymbol{\Theta} \in \mathbb{R}^{d \times d} : \operatorname{rank}(\boldsymbol{\Theta}) \le r, \|\boldsymbol{\Theta}\|_{\max} \le \alpha^* \right\}.$$
(4.24)

Theorem 14. Given n, d, r, α^* . For some underlying $\Theta \in \mathscr{K}(r, \alpha^*)$, we suppose the data $(\mathbf{X}_k, \dot{\mathbf{Y}}_k)_{k=1}^n$ are generated as in Theorem 13, i.e., truncation, dithering and quantization with parameters (4.20), and we assume $\gamma > \eta > \alpha^*$. Consider any algorithm which, for any underlying $\Theta \in \mathscr{K}(r, \alpha^*)$, takes the corresponding $(\mathbf{X}_k, \dot{\mathbf{Y}}_k)_{k=1}^n$ as input and returns $\widehat{\Theta}$. Then for some absolute constants D_1, D_2 , there exists $\Theta_0 \in \mathscr{K}(r, \alpha^*)$ such that with probability at least $\frac{3}{4}$,

$$\|\widehat{\Theta} - \Theta_0\|_F^2 / d^2 \ge \min \Big\{ D_1 \alpha^*, D_2((\alpha^*)^2 + M)r \sqrt{\frac{d}{n \log d}} \Big\}.$$
(4.25)

In the above lower bound, of primary interest is the second term — it dominates the first term in the regular scaling $n \gtrsim r^2 d$, and nearly matches the upper bound of Theorem 13 under q = 0, up to a factor of log d. Recall that the error rate in Theorem 13 suffers from essential degradation compared to the sub-Gaussian case. While the lower bound indicates that, if one only has access to $(\mathbf{X}_k, \dot{Y}_k)_{k=1}^n$ (produced by our quantization scheme with the chosen parameters), the upper bound is indeed almost tight. In other words, there exists no estimator as a function of $(\mathbf{X}_k, \dot{Y}_k)_{k=1}^n$ that could achieve error rate significantly faster than $\widehat{\Theta}$ in Theorem 13.

The proof of Theorem 14 is information-theoretic and inspired by [34, Theorem 3], but requires some modifications because of different parameter sets and sampling schemes. Similarly to Theorem 11, the lifted lower bound mainly stems from the dithering scale (4.20) that is larger than the sub-Gaussian counterpart (4.12).

To close this section, we point out that our method for 1-bit matrix completion is new and essentially different from the existing likelihood approach (see, e.g., [23, 34]). Notably, our method can deal with unknown pre-quantization noise ϵ_k that can be sub-Gaussian or heavy-tailed; note that such unknown noise precludes the standard likelihood approach. A review of prior works and more detailed comparison can be found in Appendix D.

5 An Overview of the Techniques

While deferring all the proofs to appendices, we provide an overview of the techniques used in this work. We focus on the derivation of upper bounds. We detail the sub-Gaussian regime, and use concrete example for heavy-tailed case to illustrate that the same technicalities can derive the presented results with an optimal choice of parameters. Finally, we compare our work with [45] to illustrate the main technical reason why 1-bit quantization of heavy-tailed data leads to rate degradation.

5.1 Sub-Gaussian Case

For sparse covariance matrix estimation, the element-wise error rate of Σ in Theorem 1 is a fundamental element. Unlike the full data case where $\mathbb{E}(X_{k,i}X_{k,j}) = \sigma_{ij}^*$, $\mathbb{E}\check{\sigma}_{ij} = \sigma_{ij}^*$ may not hold due to the possibility of $|X_{k,i}| > \gamma$. Thus, we first divide the element-wise error into a concentration term R_1 and a bias term R_2

$$|\breve{\sigma}_{ij} - \sigma_{ij}^*| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| := R_1 + R_2.$$

Since the quantized data is bounded, a fast concentration rate for R_1 is guaranteed by Hoeffding's inequality, while R_2 can be controlled by standard arguments. We strike a balance between R_1 , R_2 by setting $\gamma = O(\sqrt{\log(\frac{n}{\log d})})$, then the concentration term $R_1 = O(\sqrt{\frac{\log d(\log n)^2}{n}})$ dominates the error, hence the error bound only degrades by a factor $\log n$ compared with $O(\sqrt{\frac{\log d}{n}})$ for the full-data sample covariance matrix.

Recall that our estimator $\widehat{\Sigma}$ is defined by element-wisely hard thresholding $\widecheck{\Sigma}$, and the procedures to show operator norm error rate of $\widehat{\Sigma}$ are parallel to corresponding results for the full-data-based hard thresholding estimator $\mathcal{T}_{\zeta}\left(\sum_{k=1}^{n} X_k X_k^T/n\right)$ in [20]. In brief, some discussions unfold the element-wise rate $|\widehat{\sigma}_{ij} - \sigma_{ij}^*| = O\left(\min\{|\sigma_{ij}^*|, \sqrt{\frac{\log d(\log n)^2}{n}}\}\right)$, which is tighter than the bound for $|\widecheck{\sigma}_{ij} - \sigma_{ij}^*|$. This tighter rate, together with the sparsity, can yield a dimension-free bound for the dominating term of operator norm error. Despite a similar proof strategy, we need more involved analyses to deal with some new challenges from the data quantization. These additional efforts, for example, can be seen in the treatment of R_2 (A.8).

For sparse linear regression (including 1-bit QC-CS, 1-bit CS) and matrix completion, we derive the error rates for each problem based on Lemma 2, a framework of trace regression. Compared with the key lemma (Theorem 1) in [45], we present Lemma 2 in a more general form that accommodates generalized quadratic loss (3.4), and the purpose is that more flexible \boldsymbol{Q} , \boldsymbol{B} constructed from the binary data can be used. The advantage of using such framework is a rather clear proof roadmap constituted by two steps:

- Step 1. Bound $\| \max(\mathbf{Q} \cdot \operatorname{vec}(\mathbf{\Theta}^*)) \mathbf{B} \|_{\text{op}}$ from above and choose λ that guarantee (3.5);
- Step 2. Establish the restricted strong convexity (3.7), and invoke (3.8) to obtain the error rate.

We first discuss Step 1. In sparse linear regression we need $\lambda \geq 2 \| \mathbf{Q} \Theta^* - B \|_{\text{max}}$ with some \mathbf{Q}, B approximating $\Sigma_{XX} = \mathbb{E} X_k X_k^T$, $\Sigma_{YX} = \mathbb{E} Y_k X_k$, respectively. Thus, by noting $\Sigma_{XX} \Theta^* = \Sigma_{YX}$ it can be divided as two approximation error terms

$$\|\boldsymbol{Q}\Theta^* - B\|_{\max} \leq \underbrace{\|(\boldsymbol{Q} - \boldsymbol{\Sigma}_{XX})\Theta^*\|_{\max}}_{\text{approximation term I}} + \underbrace{\|B - \boldsymbol{\Sigma}_{YX}\|_{\max}}_{\text{approximation term II}}.$$

One possibility to control the approximation error term is via existing results. For instance, in 1-bit QC-CS we set Q to be the proposed sparse covariance matrix estimator $\hat{\Sigma}$. Thus, the bound of term I follows from results in Section 2 (see, e.g., (B.11)). On the other hand, we can also adopt a standard strategy of bounding the concentration error and the deviation (i.e., bias). For example, we can divide term II into (see, e.g., R_2 , R_3 in (B.20))

$$||B - \Sigma_{YX}||_{\max} \leq \underbrace{||B - \mathbb{E}B||_{\max}}_{\text{concentration term II.1}} + \underbrace{||\mathbb{E}(B - Y_k X_k)||_{\max}}_{\text{bias term II.2}}.$$

For matrix completion the methodology is similar, see (C.1) for example. We apply various concentration inequalities to bound the concentration terms, to name a few, Bernstein's inequality (B.16), (B.17), matrix Bernstein's inequality (C.2). In contrast, more standard tools

like Cauchy-Schwarz, Markov's inequality can upper bound bias terms. Let λ_{full} denote the optimal choice of λ in the full-data settings (see, e.g., [28,45,70,71]). As it comes out, in the sub-Gaussian regime of our 1-bit setting, one can always strike an almost perfect balance among all the terms such that $\lambda = \text{Poly}(\log n) \cdot \lambda_{\text{full}}$ can guarantee $\lambda \geq 2 \| \max(\boldsymbol{Q} \cdot \operatorname{vec}(\boldsymbol{\Theta}^*)) - \boldsymbol{B} \|_{\text{op}}$ (3.5).

Step 2 concerns the restricted strong convexity of Q with regard to $\widehat{\Delta}$. Note that this mainly hinges on the covariate. Thus, in 1-bit CS and matrix completion where full covariate is available, we can directly borrow existing results from the full-data settings with no quantization [28,45]. For 1-bit QC-CS with quantized X_k , the desired RSC property straightforwardly follows from Σ_{XX} 's sparsity and the resulting dimension-free bound of $\|\widehat{\Sigma} - \Sigma_{XX}\|_{op}$. Finally, we apply (3.8) to obtain the error rates. Since $\lambda = \text{Poly}(\log n) \cdot \lambda_{\text{full}}$ suffices for Step 1, under the dithered 1-bit quantization scheme, the error rate at worst degrades by logarithmic factor.

5.2 Heavy-tailed Case

In heavy-tailed regime we introduce truncation parameter η and require $\gamma > \eta$. The strategies and technical tools for the proofs are almost the same as sub-Gaussian regime, while the difference is that we can no longer strike a perfect balance among all terms. We briefly give an example to demonstrate the proofs.

Example 1. (*Theorem 10.*) To our best knowledge, Theorem 10 presents the first computationally efficient method for 1-bit CS with heavy-tailed sensing vectors, and the rate $O\left(\sqrt{s}\sqrt[3]{\frac{\log d}{n}}\right)$ (for s-sparse Θ^*) is still faster than the convex approach in [40] that is only for sub-Gaussian regime. Let us start from *Step 1* and first decompose $\|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$ into four terms (see notations given in (3.29))

$$\begin{split} \|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^{*} - \widehat{\Sigma}_{YX}\|_{\max} &\leq \|(\widehat{\Sigma}_{\tilde{X}\tilde{X}} - \Sigma_{XX})\Theta^{*}\|_{\max} + \|\widehat{\Sigma}_{YX} - \Sigma_{YX}\|_{\max} \coloneqq I + II \\ &\leq \underbrace{\|(\widehat{\Sigma}_{\tilde{X}\tilde{X}} - \mathbb{E}\widetilde{X}_{k}\widetilde{X}_{k}^{T})\Theta^{*}\|_{\max}}_{\text{concentration term I.1}} + \underbrace{\|\mathbb{E}(X_{k}X_{k}^{T} - \widetilde{X}_{k}\widetilde{X}_{k}^{T})\Theta^{*}\|_{\max}}_{\text{bias term I.2}} \\ &+ \underbrace{\|\widehat{\Sigma}_{YX} - \mathbb{E}(\gamma \cdot \dot{Y}_{k}\widetilde{X}_{k})\|_{\max}}_{\text{concentration term II.1}} + \underbrace{\|\mathbb{E}(\gamma \cdot \dot{Y}_{k}\widetilde{X}_{k} - Y_{k}X_{k})\|_{\max}}_{\text{bias term II.2}}. \end{split}$$

For two concentration terms, Bernstein's inequality gives I.1 = $O\left(\sqrt{\frac{\log d}{n}} + \eta_X^2 \frac{\log d}{n}\right)$ and II.1 = $O\left(\gamma \sqrt{\frac{\log d}{n}} + \frac{\gamma \cdot \eta_X \cdot \log d}{n}\right)$ with probability $1 - d^{-\Omega(1)}$. For two bias terms, some probability arguments and bounded 4-th moment can yield I.2 = $O\left(\frac{1}{\eta_X^2}\right)$ and II.2 = $O\left(\frac{1}{\eta_Y^2} + \frac{1}{\eta_X^2}\right)$. Recall that the heavy-tailed Y_k would be quantized to 1-bit, and we require $\gamma > \eta_Y$. To achieve an optimal trade-off among η_X, η_Y, γ , we set $\eta_X \asymp \left(\frac{n}{\log d}\right)^{1/4}, \eta_Y, \gamma \asymp \left(\frac{n}{\log d}\right)^{1/6}$, which gives an overall upper bound $\|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max} = O\left(\sqrt[3]{\frac{\log d}{n}}\right)$. Hence, $\lambda \asymp \sqrt[3]{\frac{\log d}{n}}$ suffices for $\lambda \ge 2\|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$.

For Step 2, since the truncated sample covariance matrix $\widehat{\Sigma}_{\tilde{X}\tilde{X}}$ also serves as a plug-in estimator for sparse linear regression in [45], we can directly borrow their Lemma 2(b). It should be pointed out that if we treat \widetilde{X}_k as data bounded by η_X and deal with I.1, II.1 via

Hoeffding's inequality, we can only establish an essentially slower error rate. By contrast, Bernstein's inequality enables us to make full use of X_k 's bounded 4-th moment and derive tighter bound. \triangle

5.3 Comparison With the Heavy-tailed Full-data Case

Finally, we compare the heavy-tailed, full-data setting in [45] and our heavy-tailed, 1-bit quantized setting, to explain the main technical reason why near optimal rates are derived in [45], but ours are essentially slower. The key difference is on the effectiveness of the original moment constraint. Because the truncated data admits the same moment constraint as the original data (e.g., $\mathbb{E}|\tilde{Y}_k|^4 \leq \mathbb{E}|Y_k|^4 \leq M$), [45] can use (matrix) Bernstein's inequality to deal with the concentration term. Nevertheless, quantizing \tilde{Y}_k to its 1-bit surrogate $\gamma \dot{Y}_k$ ruins the moment constraint since $\mathbb{E}|\gamma \cdot \dot{Y}_k|^4 = \gamma^4$. As a consequent, we can only derive a looser bound for the concentration term.

We use sparse linear regression as a concrete example for illustration.

Example 2. (sparse linear regression in [45] and this work.) In the proof of [45, Lemma 1], Bernstein's inequality is used to deal with the concentration term $\|\frac{1}{n}\sum_{k=1}^{n} \widetilde{Y}_{k}\widetilde{X}_{k} - \mathbb{E}\widetilde{Y}_{k}\widetilde{X}_{k}\|_{\max}$. Thanks to the moment constraints of \widetilde{X}_{k} , \widetilde{Y}_{k} , they can show

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\widetilde{Y}_{k}\widetilde{X}_{k} - \mathbb{E}\widetilde{Y}_{k}\widetilde{X}_{k}\right\|_{\max} = O\left(\sqrt{\frac{\log d}{n}} + \frac{\eta_{X}\eta_{Y}\log d}{n}\right)$$
(5.1)

with high probability. By contrast, in our Theorem 10 for 1-bit CS, the corresponding term is the concentration term II.1 in Example 1. Since $\gamma \cdot \dot{Y}_k$ fails to inherit the moment constraint from Y_k , the same Bernstein's inequality only delivers (see (B.21), (B.22))

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\gamma\cdot\dot{Y}_{k}\widetilde{X}_{k} - \mathbb{E}\gamma\cdot\dot{Y}_{k}\widetilde{X}_{k}\right\|_{\max} = O\left(\gamma\left(\sqrt{\frac{\log d}{n}} + \frac{\eta_{X}\log d}{n}\right)\right)$$
(5.2)

with probability $1 - d^{-\Omega(1)}$, which is worse since γ becomes a common factor. Furthermore, in our Theorem 8 for 1-bit QC-CS the corresponding concentration term is $\|\frac{\gamma^2}{n}\sum_{k=1}^n \dot{Y}_k \dot{X}_k - \mathbb{E}\gamma^2 \cdot \dot{Y}_k \dot{X}_k\|_{\text{max}}$. Note that both covariate and response are quantized and hence lose the moment constraint. Thus, we directly invoke Hoeffding's inequality and obtain (see (B.14), (A.12))

$$\left\|\frac{\gamma^2}{n}\sum_{k=1}^n \dot{Y}_k \dot{X}_k - \mathbb{E}\gamma^2 \cdot \dot{Y}_k \dot{X}_k\right\|_{\max} = O\left(\gamma^2 \sqrt{\frac{\log d}{n}}\right)$$
(5.3)

where γ^2 appears as a leading multiplicative factor. It shall be clear that γ or γ^2 appearing as a multiplicative factor of $\sqrt{\frac{\log d}{n}}$ leads to essential degradation.

6 Experimental Results

In this section we present experimental results on synthetic data that can corroborate and demonstrate our theories. To facilitate the presentation flow, the simulation details and the algorithms are provided in Appendix E.

6.1 Sparse Covariance Matrix Estimation

In our simulation Σ^* has exactly s-sparse columns. In sub-Gaussian regime, with high probability Theorem 3 provides the error bound

$$\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim s \sqrt{\frac{(\log n)^2 \log d}{n}}.$$
(6.1)

Thus, the operator norm error is expected to only logarithmically depends on the ambient dimension d, while essentially depend on the the sparsity s (that can be viewed as the intrinsic dimension of the problem). We draw X_k from multivariate Gaussian distribution to verify the theory. Specifically, we try (d, s) = (2500, 3), (2700, 3), (2700, 9), and test the sample size n = 900: 300: 2700 for each (d, s). The log-log error curves for all (d, s) are plotted on the left of Figure 1, with the theoretical curve $O(\frac{\log n}{\sqrt{n}})$ also provided for comparison of the error rate. Clearly, the curves with different dimension d but the same sparsity s are almost coincident, which confirms the inessential dependency on d for the error. On the other hand, the estimation error depends on s non-trivially since the curve of s = 9 is obviously higher. Moreover, the experimental curves are roughly parallel to the theoretical one, confirming a near optimal decreasing rate of $n^{-1/2}$.

In heavy-tailed regime, for Σ^* with s-sparse columns Theorem 6 guarantees

$$\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}} \lesssim s \left(\frac{\log d}{n}\right)^{1/4}.$$
(6.2)

The relation between estimation error and parameters s, d are similar to (6.1), while the convergence rate becomes slower. In our simulations, heavy-tailed data are drawn from Student's t distribution. We test (d, s) = (2200, 3), (2400, 3), (2600, 3), (2400, 9) under sample size n = 900: 300: 2400. We report the results in the right figure of Figure 1. Consistent with the error bound, three curves with same s but different d are fairly close, while larger s (s = 9) leads to essentially larger error. Although our theoretical rate $O(n^{-1/4})$ does not match the optimal rate in the classical setting, these curves seem well aligned with the theoretical curve. Furthermore, we test (d, s) = (2400, 9) with the truncation step removed and then show the error curve with legend "no truncation". One shall see the estimation error becomes worse without truncation. Therefore, truncation is not merely of technical importance, but can indeed lower the estimation error in heavy-tailed regime.

6.2 Sparse Linear Regression

1-bit quantized-covariate compressed sensing (1-bit QC-CS). In 1-bit QC-CS, both covariate X_k and response Y_k are quantized to 1-bit. Note that we use exactly sparse Θ^* , hence in sub-Gaussian regime Theorem 6 delivers the guarantee

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \log n \sqrt{\frac{s \log d}{n}},\tag{6.3}$$

while for heavy-tailed regime the error bound in Theorem 7 reads as

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \left(\frac{\log d}{n}\right)^{1/4}.$$
(6.4)



Figure 1: Sparse covariance matrix estimation. Left: Sub-Gaussian; Right: Heavy-tailed.

With simulation details given in Appendix E, we try Θ^* with (d, s) = (2400, 3), (2200, 6), (2400, 6), (2600, 6) under n = 900 : 300 : 2400. The experimental results in sub-Gaussian regime, heavy-tailed regime are shown as log-log curves on the left, the right of Figure 2, respectively. We also plot the theoretical rates for comparison. To show the efficacy of truncation in heavy-tailed regime, keeping other parameters unchanged, we test (d, s) = (2400, 3) without truncation step. The errors are accordingly shown as a curve with legend "no quantization".

The results corroborate the theory from several respects. Firstly, the curves with the same s but different d are extremely close, while the errors under s = 6 are significantly larger than s = 3. This verifies (6.4) and (6.7) that exhibit non-trivial dependence on the sparsity s but only logarithmic dependence on the ambient dimension d. Secondly, since the experimental curves are fairly aligned with the theoretical ones, the theoretical convergence rates (regarding n) are verified. Moreover, comparing the curve of (d, s) = (2400, 3) and "no quantization" on the right of Figure 2, shrinking heavy-tailed data indeed leads to more accurate estimation of Θ^* .

1-bit compressed sensing (1-bit CS). Different from the novel setting of 1-bit QC-CS, in 1-bit CS one has full covariate X_k and only quantize Y_k to 1-bit. Under *s*-sparse Θ^* , Theorem 9 gives the near minimax error bound for sub-Gaussian regime

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{\frac{s \log d \log n}{n}},\tag{6.5}$$

while Theorem 13 for heavy-tailed regime provides

$$\|\widehat{\Theta} - \Theta^*\|_2 \lesssim \sqrt{s} \left(\frac{\log d}{n}\right)^{\frac{1}{3}}.$$
(6.6)

Under sample size n = 900: 300: 2400, we test (d, s) = (2400, 3), (2200, 9), (2400, 9), (2600, 9)for sub-Gaussian case, while (d, s) = (2400, 3), (2200, 6), (2400, 6), (2600, 6) for heavy-tailed case. For (d, s) = (2400, 3) with heavy-tailed data, we also conduct an independent simulation with the truncation of X_k , Y_k removed but other conditions unchanged. The log-log error curves and the theoretical rates are plotted in Figure 3. The key implications of Figure 3 are similar to those in Figure 2 and can support our theoretical error bounds (6.5), (6.6).



Figure 2: 1-bit QC-CS. Left: Sub-Gaussian; Right: Heavy-tailed.



Figure 3: 1-bit CS. Left: Sub-Gaussian; Right: Heavy-tailed.

6.3 Low-rank Matrix Completion

While the error bounds in Theorems 12-13 are stated under $\frac{\|\widehat{\Delta}\|_{\mathrm{F}}^2}{d^2}$, we first adapt them to our simulation (see Appendix E) where the underlying exactly low-rank matrices have comparable spikiness $\alpha(\Theta^*)$ and unit Frobenius norm, and the noise is moderate compared with the signal. Specifically, under sub-Gaussian ϵ_k , we can translate MSE error bound in (4.18) into

$$\|\widehat{\mathbf{\Theta}} - \mathbf{\Theta}^*\|_{\mathrm{F}} \lesssim \sqrt{\frac{rd\log d\log n}{n}},$$
(6.7)

and similarly for heavy-tailed case (4.23)

$$\|\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}^*\|_{\mathrm{F}} \lesssim \left(\frac{r^2 d \log d}{n}\right)^{1/4}.$$
(6.8)

To corroborate the theoretical error rates, we simulate the proposed 1-bit matrix completion method using Θ^* with (d, r) = (100, 1), (100, 2), (120, 2), under the sample size n = 6000: 1000 : 10000. In heavy-tailed regime, we also try (d, r) = (120, 2) with the response truncation step removed. The experimental results are plotted as log-log error curves in Figure 4.

Clearly, in both sub-Gaussian regime (left figure) and heavy-tailed regime (right figure), the errors significantly increase when either r or d becomes larger. This corroborates the

implications of (6.7), (6.8) that the estimation error essentially hinges on r and d. Moreover, the experimental curves are well aligned with the theoretical curve, hence the theoretical error rates are confirmed. Comparing two black curves of (d, r) = (120, 2) and "no quantization" in the right figure, the truncation step seems do not bring notable improvement to the recovery of Θ^* . This is perhaps because the the moderate noise $\frac{1}{250\sqrt{3}} \cdot t(\nu = 3)$ is used in the simulation, thus making the bias-and-variance trade-off less important. On the other hand, we believe a more significant advantage of using the truncation step can be observed under severer noise.



Figure 4: Low-rank matrix completion. Left: Sub-Gaussian; Right: Heavy-tailed.

7 Concluding Remarks

In this paper we propose a dithered 1-bit quantization scheme and apply it to the estimation problems of sparse covariance matrix estimation, sparse linear regression and matrix completion. While adding a uniform dithering noise prior to quantization was already seen in literature, our scheme involves a truncation step if the data are heavy-tailed. Under highdimensional scaling, our estimators from merely binary data can recover the underlying parameters fairly well. In sub-Gaussian regime, the proposed estimators achieve near minimax rates. In heavy-tailed regime, the error rates become significantly slower because of a biasand-variance trade-off. However, these results either represent the first ones under 1-bit quantization of heavy-tailed data, or already improve on prior results. Moreover, we also derive nearly matching information-theoretic lower bounds for Theorem 10, 13 (heavy-tailed setting of 1-bit compressed sensing, matrix completion), showing that the rates are actually almost tight for estimation with the observed 1-bit data.

This work also provides new developments to each of the three estimation problems. Compared to [38] that proposed the 1-bit covariance matrix estimator $\check{\Sigma}$, the results in Section 2 can be viewed as a two-fold extension, that is, extension to high-dimensional scaling (n < d)and to heavy-tailed distribution. For sparse linear regression, we first propose and study a novel complete quantization setting where both covariate and response are quantized to 1-bit (Theorems 7-8). Corresponding results for 1-bit compressed sensing (1-bit CS) are also presented (Theorems 9-10). Compared with previous results on 1-bit CS, our sensing vector can be sub-Gaussian or heavy-tailed, and other advantages include faster rate and convex recovery program, see Appendix D. In Section 4, while all existing papers for 1-bit matrix completion (1-bit MC) are in essence based on maximum likelihood estimation, our novel method can handle pre-quantization random noise with unknown distribution, see Appendix D.

We point out two open questions to close this work. The first one is concerned with the slower error rates under the 1-bit quantization of heavy-tailed data. While they are nearly tight in 1-bit CS and 1-bit MC from our binary observations (Theorems 11, 14), it is still possible to design different (1-bit) quantization schemes for heavy-tailed data that allow faster rate. The possibility includes changing parameters in our quantization scheme. In fact, the lifted lower bounds in Theorems 11, 14 are mainly due to our choices of larger dithering scale. Such choice is the price we pay for handling heavy-tailed data, but it is possible to achieve faster rate by using smaller dithering scale (of course, deriving the faster rate in this case will rely on sharper technical tool). We leave faster estimator in a heavy-tailed, 1-bit quantized case as future work. Secondly, our results in Section 3 are non-uniform, meaning that the recovery guarantee is valid for a fixed Θ^* . Note that in nonlinear compressed sensing uniform guarantee is an important aspect and still eagerly pursued (see [29,30,49,97] for recent advances). Thus, it would be a good direction to strengthen Theorems 7-10 to a uniform ones that guarantee the recovery of all (approximately) sparse signals. For this direction, we conjecture that the main technical difficulties will lie in the 1-bit quantization and the heavy-tailed data (we refer to [30, Theorem 12] for a uniform recovery guarantee under uniformly quantized heavy-tailed data).

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Proofs: Sparse Covariance Matrix Estimation Α

Proof of Lemma 1. Since X and Λ are independent, we have

$$\mathbb{E}\Big[\gamma \cdot \operatorname{sign}(X+\Lambda)\Big] = \mathbb{E}_X \mathbb{E}_{\Lambda}\Big[\gamma \cdot \operatorname{sign}(X+\Lambda)\Big] = \mathbb{E}_X\Big[\gamma \cdot \mathbb{P}(\Lambda \ge -X) + (-\gamma) \cdot \mathbb{P}(\Lambda < -X)\Big]$$
$$= \mathbb{E}_X\Big[\gamma \cdot \Big(\frac{\gamma+X}{2\gamma} - \frac{\gamma-X}{2\gamma}\Big)\Big] = \mathbb{E}X.$$

note that the third equal sign relies on $\gamma \geq B$.

Proof of Corollary 1. Since Λ_1 and Λ_2 are i.i.d. uniformly distributed on $[-\gamma, \gamma]$ and independent of X, Y, then by using Lemma 1 we have

$$\mathbb{E}\Big[\gamma^2 \cdot \operatorname{sign}(X + \Lambda_1)\operatorname{sign}(Y + \Lambda_2)\Big] = \mathbb{E}_{X,Y}\mathbb{E}_{\Lambda_1}\mathbb{E}_{\Lambda_2}\Big[\gamma^2 \cdot \operatorname{sign}(X + \Lambda_1)\operatorname{sign}(Y + \Lambda_2)\Big]$$
$$= \mathbb{E}_{X,Y}\Big(\mathbb{E}_{\Lambda_1}\Big[\gamma \cdot \operatorname{sign}(X + \Lambda_1)\Big]\mathbb{E}_{\Lambda_2}\Big[\gamma \cdot \operatorname{sign}(Y + \Lambda_2)\Big]\Big) = \mathbb{E}XY,$$
esult follows.

the result follows.

Sub-Gaussian Data A.1

Proof of Theorem 1. For fixed i, j, triangle inequality yields

$$|\breve{\sigma}_{ij} - \sigma_{ij}^*| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| := R_1 + R_2.$$
(A.1)

It suffices to bound R_1 , R_2 from above.

<u>Bound of R_1 </u>. We introduce the element-wise notation of the quantized data as \dot{X}_{kj} $\begin{aligned} \overline{[\dot{X}_{kj,1}, \dot{X}_{kj,2}, \dots, \dot{X}_{kj,d}]^T}, \ \forall \ k \in [n], \ j \in [2], \ \text{then by} \ (2.1) \ \breve{\sigma}_{ij} &= \frac{1}{n} \sum_{k=1}^n \frac{\gamma^2}{2} \left[\dot{X}_{k1,i} \dot{X}_{k2,j} + \dot{X}_{k2,i} \dot{X}_{k1,j} \right]. \\ \text{Since} \ \left| \frac{\gamma^2}{2} \left[\dot{X}_{k1,i} \dot{X}_{k2,j} + \dot{X}_{k2,i} \dot{X}_{k1,j} \right] \right| &\leq \gamma^2, \ \text{Hoeffding's inequality} \ (\text{Proposition 3}) \ \text{yields} \end{aligned}$

$$\mathbb{P}(|\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \ge t) \le 2\exp(-nt^2/2\gamma^4), \ \forall \ t > 0.$$

We set $t = \gamma^2 \sqrt{\frac{2\delta \log d}{n}}$ and obtain

$$\mathbb{P}\left(R_1 \ge \gamma^2 \sqrt{\frac{2\delta \log d}{n}}\right) \le 2d^{-\delta}.$$
(A.2)

Bound of R_2 . By Corollary 1 and some algebra, we have

$$R_{2} = \left| \mathbb{E} \left(\gamma^{2} \cdot \dot{X}_{k1,i} \dot{X}_{k2,j} - X_{k,i} X_{k,j} \right) \right|$$

= $\left| \mathbb{E} \left[\gamma^{2} \dot{X}_{k1,i} \dot{X}_{k2,j} - X_{k,i} X_{k,j} \right] \left[\mathbb{1} \left(\{ |X_{k,i}| \ge \gamma \} \cup \{ |X_{k,j}| > \gamma \} \right) \right] \right|$
 $\leq \mathbb{E} \left| X_{k,i} X_{k,j} \right| \mathbb{1} \left(|X_{k,i}| > \gamma \right) + \mathbb{E} \left| X_{k,i} X_{k,j} \right| \mathbb{1} \left(|X_{k,j}| > \gamma \right) := R_{21} + R_{22}$

Note that R_{21} , R_{22} can be bounded likewise, thus we only show the upper bound of R_{21} . We use Cauchy-Schwarz inequality, and then Proposition 1, it yields

$$R_{21} \leq \sqrt{\mathbb{E}|X_{k,i}X_{k,j}|^2} \cdot \sqrt{\mathbb{P}(|X_{k,i}| > \gamma)} \leq \sqrt{\frac{1}{2}\mathbb{E}(|X_{k,i}|^4 + |X_{k,j}|^4)} \cdot \sqrt{\mathbb{P}(|X_{k,i}| > \gamma)}$$
$$\lesssim \sqrt{\sigma^4} \cdot \sqrt{\exp\left(-\frac{D_1\gamma^2}{\sigma^2}\right)} \leq \sigma^2 \exp\left(-\frac{D_1\gamma^2}{2\sigma^2}\right).$$

We further plug in (2.6) and assume C_1 is sufficiently large such that $D_1 C_1^2 \ge 1$, it delivers $R_{21} \lesssim \sigma^2 \sqrt{\frac{2\delta \log d}{n}}$. Therefore, we conclude that

$$R_2 = |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| \lesssim \sigma^2 \sqrt{\frac{2\delta \log d}{n}}.$$
(A.3)

Combining (A.2) and (A.3) we derive $\mathbb{P}\left(|\breve{\sigma}_{ij} - \sigma^*_{ij}| \lesssim \gamma^2 \sqrt{\frac{\delta \log d}{n}}\right) \ge 1 - 2d^{\delta}$. With no loss of generality, we can assume $2\delta \log d > e$, then $\gamma^2 \lesssim \sigma^2 \log n$, then (2.7) follows. It is not hard to see that (2.8) follows from (2.7) via a union bound.

Proof of Theorem 2. Since γ has been specified with some C_1 , from Theorem 1 we know there exists an absolute constant D_1 such that

$$\mathbb{P}\Big(|\breve{\sigma}_{ij} - \sigma_{ij}^*| \le D_1 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big) \ge 1 - 2d^{-\delta}.$$
(A.4)

Assume C_2 is sufficiently large such that $C_2 > D_1$. We first rule out $2d^{-\delta}$ probability and assume $|\breve{\sigma}_{ij} - \sigma_{ij}^*| \leq D_1 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}$. Recall that $\widehat{\sigma}_{ij} = \mathcal{T}_{\zeta}(\breve{\sigma}_{ij})$, we analyse two cases. <u>Case 1.</u> $|\breve{\sigma}_{ij}| < \zeta$, then by definition we have $\widehat{\sigma}_{ij} = 0$, hence $|\widehat{\sigma}_{ij} - \sigma_{ij}^*| = |\sigma_{ij}^*| \leq |\sigma_{ij}^*|$. Besides, by triangle inequality we have $|\sigma_{ij}^*| \leq |\sigma_{ij}^* - \breve{\sigma}_{ij}| + |\breve{\sigma}_{ij}| \leq (D_1 + C_2)\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}$, hence we have

$$|\widehat{\sigma}_{ij} - \sigma_{ij}^*| \le (D_1 + C_2 + 1) \min\left\{ |\sigma_{ij}^*|, \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}} \right\}.$$

<u>Case 2.</u> $|\breve{\sigma}_{ij}| \geq \zeta$, then we have $\widehat{\sigma}_{ij} = \breve{\sigma}_{ij}$, hence $|\widehat{\sigma}_{ij} - \sigma^*_{ij}| = |\breve{\sigma}_{ij} - \sigma^*_{ij}| \leq D_1 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}$. Moreover, since $C_2 > D_1$, we have $|\sigma^*_{ij}| \geq |\breve{\sigma}_{ij}| - |\breve{\sigma}_{ij} - \sigma^*_{ij}| \geq (C_2 - D_1)\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}$, which implies that $\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}} \leq \frac{1}{C_2 - D_1} |\sigma^*_{ij}|$, hence we have $|\widehat{\sigma}_{ij} - \sigma^*_{ij}| \leq \frac{D_1}{C_2 - D_1} |\sigma^*_{ij}|$. By putting pieces together we obtain

$$|\widehat{\sigma}_{ij} - \sigma_{ij}^*| \le \left(D_1 + \frac{D_1}{C_2 - D_1}\right) \min\left\{|\sigma_{ij}^*|, \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right\}.$$

Combining two cases leads to (2.10), hence the proof is concluded.

Proof of Theorem 3. Since γ and ζ are properly set with some C_1, C_2 , by Theorem 2, (2.10) holds with some absolute constant D_1 hidden behind " \lesssim ". For convenience we define

$$\mathscr{A}_{ij} = \left\{ |\widehat{\sigma}_{ij} - \sigma_{ij}^*| \le D_1 \min\{|\sigma_{ij}^*|, \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\} \right\}.$$
 (A.5)

Let \mathscr{A}_{ij}^c be its complement, then we have $\mathbb{P}(\mathscr{A}_{ij}^c) \leq 2d^{-\delta}$. For $d \times d$ symmetric matrix $A = [\alpha_1, ..., \alpha_n]$ with columns α_j , we have $||A||_{\text{op}} \leq \sup_{j \in [d]} ||\alpha_j||_1$. Thus, some algebra gives

$$\mathbb{E}\|\widehat{\Sigma} - \Sigma^*\|_{\text{op}}^p \leq \mathbb{E}\left[\sup_{j\in[d]}\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\right]^p \leq \mathbb{E}\sup_{j\in[d]}\left[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij}) + \sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij})\right]^p \\ \leq 2^p \mathbb{E}\sup_{j\in[d]}\left[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij})\right]^p + 2^p \mathbb{E}\sup_{j\in[d]}\left[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij}^c)\right]^p := R_1 + R_2,$$
(A.6)

<u>Bound of R_1 .</u> Let us first bound R_1 . By (2.3) and (A.5) we have

$$\sum_{i=1}^{d} |\widehat{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}) \leq \sum_{i=1}^{d} D_{1} \min\left\{|\sigma_{ij}^{*}|, \sigma^{2} \log n \sqrt{\frac{\delta \log d}{n}}\right\}$$

$$\leq \sum_{i=1}^{d} D_{1} |\sigma_{ij}^{*}|^{q} \left(\sigma^{2} \log n \sqrt{\frac{\delta \log d}{n}}\right)^{1-q} \leq D_{1} s \left(\sigma^{2} \log n \sqrt{\frac{\delta \log d}{n}}\right)^{1-q}.$$
(A.7)

This further gives

$$R_1 \le \left(2D_1 s \left[\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right]^{1-q}\right)^p.$$

<u>Bound of R2.</u> Recall $\widehat{\sigma}_{ij} = \mathcal{T}_{\zeta} \breve{\sigma}_{ij}$, let $T_1 = \left[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma_{ij}^*| \mathbb{1}(\mathscr{A}_{ij}^c)\right]^p$, we have

$$T_{1} \leq \Big[\sum_{i=1}^{d} |\sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\breve{\sigma}_{ij}| < \zeta) + \sum_{i=1}^{d} |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \mathbb{1}(\mathscr{A}_{ij}^{c}) + \sum_{i=1}^{d} |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c})\Big]^{p} \\ \leq (3d)^{p-1} \Big[\sum_{i=1}^{d} |\sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\breve{\sigma}_{ij}| < \zeta) + \sum_{i=1}^{d} |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) + \sum_{i=1}^{d} |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c})\Big].$$

Combining with the form of R_2 yields

$$R_{2} \leq 2^{p} \mathbb{E} \sum_{j=1}^{d} \left[\sum_{i=1}^{d} |\widehat{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c}) \right]^{p} \leq 6^{p} d^{p-1} \left(\mathbb{E} \sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\breve{\sigma}_{ij}| < \zeta) + \mathbb{E} \sum_{i,j} |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) + \mathbb{E} \sum_{i,j} |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \right) := 2^{p} d^{p-1} \left(R_{21} + R_{22} + R_{23} \right).$$
(A.8)

Let us deal with R_{21}, R_{22}, R_{23} separately.

<u>Bound of R₂₁.</u> Suppose the event $\mathscr{A}_{ij}^c \cap \{ |\breve{\sigma}_{ij}| < \zeta \}$ holds, then $\widehat{\sigma}_{ij} = 0$, combining with (A.5) we know $|\widehat{\sigma}_{ij} - \sigma_{ij}^*| = |\sigma_{ij}^*| > D_1 \min\{ |\sigma_{ij}^*|, \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}} \}$. Recall (A.3), we assume $|\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| \le D_{1,0}\sigma^2 \sqrt{\frac{\delta \log d}{n}}$ for some constant $D_{1,0}$. To avoid technical complication, we simply assume D_1, C_2 are sufficiently large and satisfy $D_1 \ge \max\{3C_2, 3\}, C_2 \ge \max\{D_{1,0}, 10C_1^2\}$.

Combining with (2.9) in Theorem 2, we have

$$|\sigma_{ij}^*| \ge D_1 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}} \ge 3\zeta > 3|\breve{\sigma}_{ij}| \ge 3|\sigma_{ij}^*| - 3|\sigma_{ij}^* - \breve{\sigma}_{ij}|,$$

which implies

$$|\sigma_{ij}^* - \breve{\sigma}_{ij}| \ge \frac{2}{3} |\sigma_{ij}^*| \quad \text{and} \quad \zeta \le \frac{1}{3} |\sigma_{ij}^*|. \tag{A.9}$$

Moreover, we have

$$|\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| \le D_{1,0}\sigma^2 \sqrt{\frac{\delta \log d}{n}} \le \zeta \le \frac{1}{3}|\sigma_{ij}^*|.$$

Besides $|\sigma_{ij}^*| \ge 3\zeta$, based on $|\sigma_{ij}^* - \check{\sigma}_{ij}| \ge \frac{2}{3} |\sigma_{ij}^*|$, we use triangle inequality and obtain

$$\frac{2}{3}|\sigma_{ij}^*| \le |\sigma_{ij}^* - \breve{\sigma}_{ij}| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + \frac{1}{3}|\sigma_{ij}^*|,$$
(A.10)

which implies $|\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \geq \frac{1}{3}|\sigma^*_{ij}|$. Therefore, we draw the conclusion that

$$\mathscr{A}_{ij}^c \cap \{ |\breve{\sigma}_{ij}| < \zeta \} \Longrightarrow \{ |\sigma_{ij}^*| > 3\zeta \} \cap \{ |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \ge \frac{1}{3} |\sigma_{ij}^*| \}.$$

Now we can invoke Hoeffding's inequality (Proposition 3) and obtain

$$R_{21} = \sum_{i,j} |\sigma_{ij}^*|^p \mathbb{E} \left[\mathbb{1}(\mathscr{A}_{ij}^c \cap \{|\breve{\sigma}_{ij}| < \zeta\}) \right]$$

$$\leq \sum_{i,j} |\sigma_{ij}^*|^p \mathbb{1}(|\sigma_{ij}^*| > 3\zeta) \mathbb{P} \left(|\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \ge \frac{1}{3} |\sigma_{ij}^*| \right)$$

$$\leq 2 \sum_{i,j} |\sigma_{ij}^*|^p \mathbb{1}(|\sigma_{ij}^*| > 3\zeta) \exp \left(-\frac{n |\sigma_{ij}^*|^2}{18\gamma^4} \right)$$

Moreover, some calculus can verify $\sup_{y\geq 0} y^{\frac{p}{2}} \exp(-y/36) \leq (D_2)^p (\sqrt{p})^p$. Thus, we proceed as

$$\begin{aligned} R_{21} &\leq 2\sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{1}(|\sigma_{ij}^{*}| > 3\zeta) \exp\left(-\frac{n|\sigma_{ij}^{*}|^{2}}{18\gamma^{4}}\right) \\ &= 2\left(\frac{\gamma^{2}}{\sqrt{n}}\right)^{p} \sum_{i,j} \left(\left[\frac{n|\sigma_{ij}^{*}|^{2}}{\gamma^{4}}\right]^{\frac{p}{2}} \exp\left[-\frac{n|\sigma_{ij}^{*}|^{2}}{36\gamma^{4}}\right]\right) \left(\mathbb{1}(|\sigma_{ij}^{*}| > 3\zeta) \exp\left[-\frac{n|\sigma_{ij}^{*}|^{2}}{36\gamma^{4}}\right]\right) \\ &\leq 2\left(\frac{\gamma^{2}}{\sqrt{n}}\right)^{p} \left(\sup_{y\geq 0} y^{\frac{p}{2}} \exp\left[-\frac{y}{36}\right]\right) \left(d^{2} \exp\left[-\frac{n\zeta^{2}}{4\gamma^{4}}\right]\right), \end{aligned}$$

Recall (2.6), (2.9) and that we assume $C_2 \geq 10C_1^2$, we have $\zeta \geq 10C_1^2\sigma^2\log n\sqrt{\frac{\delta\log d}{n}} \geq 10\gamma^2\sqrt{\frac{\delta\log d}{n}}$, which delivers $d^2\exp(-\frac{n\zeta^2}{4\gamma^4}) \leq d^{2-25\delta}$. We now put pieces together and obtain

$$R_{21} \le d^{2-25\delta} \left(2D_2 \gamma^2 \sqrt{\frac{p}{n}} \right)^p \le d^{2-25\delta} \left(D_3 \sigma^2 \log n \sqrt{\frac{\delta}{n}} \right)^p.$$

Bound of R_{22} . By Cauchy-Schwarz inequality we have

$$R_{22} \leq \sum_{i,j} \sqrt{\mathbb{E} |\breve{\sigma}_{ij} - \mathbb{E} \breve{\sigma}_{ij}|^{2p} \mathbb{P}(\mathscr{A}_{ij}^c)} \leq \sum_{i,j} d^{-\frac{\delta}{2}} \sqrt{2\mathbb{E} |\breve{\sigma}_{ij} - \mathbb{E} \breve{\sigma}_{ij}|^{2p}}.$$

Recall $\check{\sigma}_{ij} = \frac{1}{n} \sum_{k=1}^{n} \frac{\gamma^2}{2} [\dot{X}_{k1,i} \dot{X}_{k2,j} + \dot{X}_{k2,i} \dot{X}_{k1,j}]$ with each summand lying between $[-\frac{\gamma^2}{n}, \frac{\gamma^2}{n}]$, so by Hoeffding's Lemma (e.g., Lemma 1.8 in [82]), $\check{\sigma}_{ij} - \mathbb{E}\check{\sigma}_{ij}$ is the sum of *n* independent random variable, and each variable has sub-Gaussian norm scaling $O(\frac{\gamma^2}{n})$. Thus, Proposition 2 gives $\|\check{\sigma}_{ij} - \mathbb{E}\check{\sigma}_{ij}\|_{\psi_2} = O(\frac{\gamma^2}{\sqrt{n}})$. Now we invoke Proposition 1(b) to obtain

$$R_{22} \lesssim d^{2-\frac{\delta}{2}} \left(D_{4,0} \gamma^2 \sqrt{\frac{p}{n}} \right)^p \leq d^{2-\frac{\delta}{2}} \left(D_4 \sigma^2 \log n \sqrt{\frac{\delta}{n}} \right)^p.$$

<u>Bound of R₂₃</u>. Note that $|\mathbb{E}\breve{\sigma}_{ij} - \sigma^*_{ij}|$ is constant, hence we use (A.3) and obtain

$$R_{23} \le |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*|^p \sum_{i,j} 2d^{-\delta} \le 2d^{2-\delta} \Big(D_5 \sigma^2 \sqrt{\frac{\delta \log d}{n}} \Big)^p.$$

Now we are in a position to put everything together. By combining the upper bounds for $R_{2i}, i = 1, 2, 3$, we have $R_{21} + R_{22} + R_{23} \le d^{2-\frac{\delta}{2}} \left(D_6 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}} \right)^p$. Substitute it into (A.8), recall $p = \frac{\delta}{4}$ and $\delta \ge 4$, we obtain

$$R_2 \le d^{1-\frac{\delta}{4}} \left(6D_6 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right)^p \le \left(6D_6 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right)^p.$$

This bound is dominated by the bound of R_1 when $\delta \log d(\log n)^2/n$ is sufficiently small (note that conventionally one assumes $s = \Omega(1)$). Thus, there exists absolute constant D_7 such that

$$\mathbb{E}\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}}^p \le \left(D_7 s \left[\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\right]^{1-q}\right)^p,$$

which gives (2.11). We further invoke Markov inequality:

$$\mathbb{P}\Big(\|\widehat{\Sigma} - \Sigma^*\|_{\mathrm{op}} \ge e^4 D_7 s \Big[\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big]^{1-q}\Big)$$
$$= \mathbb{P}\Big(\|\widehat{\Sigma} - \Sigma^*\|_{\mathrm{op}}^p \ge \Big(e^4 D_7 s \Big[\sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big]^{1-q}\Big)^p\Big) \le \exp(-4p) = \exp(-\delta),$$

(2.12) follows. Now the proof is concluded.

A.2 Heavy-tailed Data

Proof of Theorem 4. Since $\gamma > \eta \ge |\widetilde{X}_{k,i}|$, by using Corollary 1 we can "expect out" the independent dithering noises $\Gamma_{k1,i}, \Gamma_{k2,j}$,

$$\mathbb{E}\breve{\sigma}_{ij} = \mathbb{E}\left(\gamma^2 \cdot \operatorname{sign}(\widetilde{X}_{k,i} + \Gamma_{k1,i})\operatorname{sign}(\widetilde{X}_{k,j} + \Gamma_{k2,j})\right)$$
$$= \mathbb{E}_{\widetilde{X}_{k,i}\widetilde{X}_{k,j}}\left(\mathbb{E}_{\Gamma_{k1,i}}\left[\gamma \cdot \operatorname{sign}(\widetilde{X}_{k,i} + \Gamma_{k1,i})\right]\right)\left(\mathbb{E}_{\Gamma_{k2,j}}\left[\gamma \cdot \operatorname{sign}(\widetilde{X}_{k,j} + \Gamma_{k2,j})\right]\right) = \mathbb{E}\widetilde{X}_{k,i}\widetilde{X}_{k,j}$$

Thus, by triangle inequality we have

$$|\breve{\sigma}_{ij} - \sigma^*_{ij}| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + |\mathbb{E}(X_{k,i}X_{k,j} - \widetilde{X}_{k,i}\widetilde{X}_{k,j})| := R_1 + R_2.$$
(A.11)

<u>Bound of R1</u>. From $\breve{\sigma}_{ij} = \sum_{k=1}^{n} \frac{\gamma^2}{2n} (\dot{X}_{k1,i} \dot{X}_{k2,j} + \dot{X}_{k1,j} \dot{X}_{k2,i})$ we know $\breve{\sigma}_{ij}$ is mean of n independent random variables lying in $[-\gamma^2, \gamma^2]$, then by Hoeffding's inequality (Proposition 3) and plug in the value of γ (2.15), we have

$$\mathbb{P}(R_1 \ge t) \le 2 \exp\left(-\frac{nt^2}{2\gamma^4}\right) = 2 \exp\left(-\frac{t^2 \sqrt{n\delta \log d}}{2C_4^4 M}\right), \ \forall t > 0.$$
(A.12)

Setting $t = \sqrt{2M}C_4^2 \left(\frac{\delta \log d}{n}\right)^{1/4}$ yields $\mathbb{P}\left(R_1 \ge \sqrt{2}C_4^2 \sqrt{M}\left[\frac{\delta \log d}{n}\right]^{1/4}\right) \le 2d^{-\delta}$. Bound of R_2 . Since the truncated version $\widetilde{X}_{k,i} \ne X_{k,i}$ only when $|X_{k,i}| > \eta$, so

$$R_{2} \leq \mathbb{E} \left[|X_{k,i}X_{k,j} - \widetilde{X}_{k,i}\widetilde{X}_{k,j}| (\mathbb{1}(\{|X_{k,i}| > \eta\} \cup \{|X_{k,j}| > \eta\})) \right]$$

= $\mathbb{E} \left[|X_{k,i}X_{k,j}| \mathbb{1}(|X_{k,i}| > \eta) \right] + \mathbb{E} \left[|X_{k,i}X_{k,j}| \mathbb{1}(|X_{k,j}| > \eta) \right] := R_{21} + R_{22}$

By Cauchy-Schwarz inequality, we bound R_{21} by $R_{21} \leq \sqrt{\mathbb{E}|X_{k,i}X_{k,j}|^2\mathbb{P}(|X_{k,i}| > \eta)}$, moreover, we have $\mathbb{E}|X_{k,i}X_{k,j}|^2 \leq \mathbb{E}(|X_{k,i}|^4 + |X_{k,j}|^4)/2 \leq M$. A direct application of Markov inequality yields that $\mathbb{P}(|X_{k,i}| > \eta) \leq \frac{\mathbb{E}|X_{k,i}|^4}{\eta^4} = \frac{M}{\eta^4}$. Plug in the above two inequalities and the value of η , we have $R_{21} \leq \frac{M}{\eta^2} = \frac{\sqrt{M}}{C_3^2} \left(\frac{\delta \log d}{n}\right)^{1/4}$. Since R_{22} can be bounded likewise, it holds that

$$R_2 = \left| \mathbb{E} (X_{k,i} X_{k,j} - \widetilde{X}_{k,i} \widetilde{X}_{k,j}) \right| \le \frac{2}{C_3^2} \sqrt{M} \left(\frac{\delta \log d}{n} \right)^{\frac{1}{4}}.$$
 (A.13)

Now we can put things together and obtain (2.16). Moreover, (2.17) follows from a union bound, hence the proof is concluded. \Box

Proof of Theorem 5. The proof is parallel to that of Theorem 2. For some specified C_3, C_4 , by Theorem 4 there exists an absolute constant D_1 such that

$$\mathbb{P}\Big(|\breve{\sigma}_{ij} - \sigma_{ij}^*| \le D_1 \sqrt{M} \Big[\frac{\delta \log d}{n}\Big]^{\frac{1}{4}}\Big) \ge 1 - 2d^{-\delta}.$$
(A.14)

We assume $C_5 > D_1$ and first rule out probability $2d^{-\delta}$ in (A.14), so we can proceed the proof upon the event $|\check{\sigma}_{ij} - \sigma_{ij}^*| \leq D_1 \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}$. According to the threshold ζ we discuss two cases.

<u>Case 1.</u> $|\breve{\sigma}_{ij}| < \zeta$, then we have $\widehat{\sigma}_{ij} = 0$, thus, $|\widehat{\sigma}_{ij} - \sigma^*_{ij}| = |\sigma^*_{ij}| \le |\sigma^*_{ij}|$. Moreover, triangle inequality gives $|\sigma^*_{ij}| \le |\sigma^*_{ij} - \breve{\sigma}_{ij}| + |\breve{\sigma}_{ij}| \le (D_1 + C_5)\sqrt{M} \left(\frac{\delta \log d}{n}\right)^{\frac{1}{4}}$, so we have

$$\left|\widehat{\sigma}_{ij} - \sigma_{ij}^*\right| \le (D_1 + C_5 + 1) \min\left\{\left|\sigma_{ij}^*\right|, \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{\frac{1}{4}}\right\}$$

<u>Case 2.</u> $|\check{\sigma}_{ij}| > \zeta$, then we have $\widehat{\sigma}_{ij} = \check{\sigma}_{ij}$, which leads to $|\widehat{\sigma}_{ij} - \sigma^*_{ij}| = |\check{\sigma}_{ij} - \sigma^*_{ij}| \le D_1 \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}$. Let us show it can also be bounded by $|\sigma^*_{ij}|$. A reverse triangle inequality gives

$$|\sigma_{ij}^*| \ge |\breve{\sigma}_{ij}| - |\breve{\sigma}_{ij} - \sigma^*| > \zeta - |\breve{\sigma}_{ij} - \sigma^*| \ge (C_5 - D_1)\sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4},$$

so we obtain $\sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4} \leq \frac{1}{C_5 - D_1} |\sigma_{ij}^*|$. Now we can draw the conclusion that

$$|\widehat{\sigma}_{ij} - \sigma_{ij}^*| \le (D_1 + \frac{D_1}{C_5 - D_1}) \min\left\{ |\sigma_{ij}^*|, \sqrt{M} \left[\frac{\delta \log d}{n} \right]^{\frac{1}{4}} \right\}.$$

Combining two cases leads to (2.20), so we complete the proof.

Proof of Theorem 6. Since η, γ, ζ are specified with some C_3, C_4, C_5 , by Theorem 5 there exists absolute constant D_1 such that (2.20) holds. We define the event

$$\mathscr{A}_{ij} = \left\{ |\widehat{\sigma}_{ij} - \sigma^*_{ij}| \le D_1 \min\{|\sigma^*_{ij}|, \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{\frac{1}{4}} \} \right\},\tag{A.15}$$

then we have $\mathbb{P}(\mathscr{A}_{ij}^c) \leq 2d^{-\delta}$ (Here, \mathscr{A}_{ij}^c denotes the complementary event). Now we can divide the operator norm error according to \mathscr{A}_{ij} and \mathscr{A}_{ij}^c , it gives

$$\mathbb{E}\|\widehat{\Sigma} - \Sigma^*\|_{\mathrm{op}}^p \leq \mathbb{E}\Big[\sup_{j\in[d]}\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\Big]^p \leq \mathbb{E}\sup_{j\in[d]}\Big[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij}) + \sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij})\Big]^p$$
$$\leq 2^p \mathbb{E}\sup_{j\in[d]}\Big[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij})\Big]^p + 2^p \mathbb{E}\sup_{j\in[d]}\Big[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma^*_{ij}|\mathbb{1}(\mathscr{A}_{ij})\Big]^p := R_1 + R_2.$$

Bound of R_1 . By the sparsity (2.3) and (A.15), for any $j \in [d]$ we have

$$\sum_{i=1}^{d} |\widehat{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}) \leq \sum_{i=1}^{d} D_{1} \min\left\{|\sigma_{ij}^{*}|, \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{\frac{1}{4}}\right\}$$
$$\leq \sum_{i=1}^{d} D_{1} |\sigma_{ij}^{*}|^{q} \left(\sqrt{M} \left[\frac{\delta \log d}{n}\right]^{\frac{1}{4}}\right)^{1-q} \leq D_{1} s \left(\sqrt{M} \left[\frac{\delta \log d}{n}\right]^{\frac{1}{4}}\right)^{1-q}.$$

This leads to

$$R_1 \le \left(2D_1 s M^{(1-q)/2} \left(\frac{\delta \log d}{n}\right)^{(1-q)/4}\right)^p.$$

<u>Bound of R₂.</u> Let $T_1 = \left[\sum_{i=1}^d |\widehat{\sigma}_{ij} - \sigma_{ij}^*| \mathbb{1}(\mathscr{A}_{ij}^c)\right]^p$. Recall that $\widehat{\sigma}_{ij} = \mathcal{T}_{\zeta} \check{\sigma}_{ij}$, under \mathscr{A}_{ij}^c we divide the problem into $\{|\check{\sigma}_{ij}| < \zeta\}$ and $\{|\check{\sigma}_{ij}| \geq \zeta\}$, then triangle inequality yields

$$T_{1} \leq \left[\sum_{i=1}^{d} |\sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\breve{\sigma}_{ij}| < \zeta) + \sum_{i=1}^{d} |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \mathbb{1}(\mathscr{A}_{ij}^{c}) + \sum_{i=1}^{d} |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c})\right]^{p}$$
$$\leq (3d)^{p-1} \left[\sum_{i=1}^{d} |\sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\breve{\sigma}_{ij}| < \zeta) + \sum_{i=1}^{d} |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) + \sum_{i=1}^{d} |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c})\right].$$

Now we put it into the expression of R_2 and obtain

$$R_{2} \leq 2^{p} \mathbb{E} \sum_{j=1}^{d} \left[\sum_{i=1}^{d} |\widehat{\sigma}_{ij} - \sigma_{ij}^{*}| \mathbb{1}(\mathscr{A}_{ij}^{c}) \right]^{p} \leq 6^{p} d^{p-1} \left(\mathbb{E} \sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \mathbb{1}(|\check{\sigma}_{ij}| < \zeta) + \mathbb{E} \sum_{i,j} |\check{\sigma}_{ij} - \mathbb{E}\check{\sigma}_{ij}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) + \mathbb{E} \sum_{i,j} |\mathbb{E}\check{\sigma}_{ij} - \sigma_{ij}^{*}|^{p} \mathbb{1}(\mathscr{A}_{ij}^{c}) \right) := 6^{p} d^{p-1} (R_{21} + R_{22} + R_{23}).$$
(A.16)

<u>Bound of R₂₁</u>. Suppose the event $\mathscr{A}_{ij}^c \cap \{|\check{\sigma}_{ij}| < \zeta\}$ holds, then $\widehat{\sigma}_{ij} = 0$, thus, (A.15) delivers that $|\widehat{\sigma}_{ij} - \sigma_{ij}^*| = |\sigma_{ij}^*| > D_1 \min\{|\sigma_{ij}^*|, \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}\}$. With no loss of generality, we assume $D_1 \ge \max\{3C_5, 3\}$, and $C_5 \ge \max\{2/C_3^2, 4C_4^2\}$. Combining with (2.19), Theorem 5, we have

$$|\sigma_{ij}^*| \ge D_1 \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4} \ge 3\zeta > 3|\breve{\sigma}_{ij}| \ge 3|\sigma_{ij}^*| - 3|\sigma_{ij}^* - \breve{\sigma}_{ij}|,$$

which implies $|\sigma_{ij}^* - \breve{\sigma}_{ij}| \ge \frac{2}{3} |\sigma_{ij}^*|$ and $\zeta \le \frac{1}{3} |\sigma_{ij}^*|$. Since $\eta < \gamma$, it always holds that $\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^* = \mathbb{E}(X_{k,i}X_{k,j} - \widetilde{X}_{k,i}\widetilde{X}_{k,j})$. Combining (A.13), (2.19) gives

$$\left|\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*\right| \le \frac{2}{C_3^2} \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4} \le C_5 \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4} \le \zeta \le \frac{1}{3} |\sigma_{ij}^*|.$$

We upper bound $|\sigma_{ij}^* - \breve{\sigma}_{ij}|$ by triangle inequality and have

$$\frac{2}{3}|\sigma_{ij}^*| \le |\sigma_{ij}^* - \breve{\sigma}_{ij}| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*| \le |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| + \frac{1}{3}|\sigma_{ij}^*|,$$

which implies $|\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \geq \frac{1}{3}|\sigma_{ij}^*|$. Therefore,

$$\mathscr{A}_{ij}^c \cap \{ |\breve{\sigma}_{ij}| < \zeta \} \Longrightarrow \{ |\sigma_{ij}^*| > 3\zeta \} \cap \{ |\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \ge \frac{1}{3} |\sigma_{ij}^*| \},\$$

so we can bound R_{21} via

$$\begin{split} R_{21} &= \sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{E} \Big[\mathbb{1}(\mathscr{A}_{ij}^{c} \cap \{ |\breve{\sigma}_{ij}| < \zeta \}) \Big] \\ &\leq \sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{1}(|\sigma_{ij}^{*}| > 3\zeta) \mathbb{P} \Big(|\breve{\sigma}_{ij} - \mathbb{E}\breve{\sigma}_{ij}| \ge \frac{1}{3} |\sigma_{ij}^{*}| \Big) \le 2 \sum_{i,j} |\sigma_{ij}^{*}|^{p} \mathbb{1}(|\sigma_{ij}^{*}| > 3\zeta) \exp \Big(-\frac{n |\sigma_{ij}^{*}|^{2}}{18\gamma^{4}} \Big) \\ &= 2 \Big(\frac{\gamma^{2}}{\sqrt{n}} \Big)^{p} \sum_{i,j} \Big(\Big[\frac{n |\sigma_{ij}^{*}|^{2}}{\gamma^{4}} \Big]^{\frac{p}{2}} \exp \Big[-\frac{n |\sigma_{ij}^{*}|^{2}}{36\gamma^{4}} \Big] \Big) \Big(\mathbb{1}(|\sigma_{ij}^{*}| > 3\zeta) \exp \Big[-\frac{n |\sigma_{ij}^{*}|^{2}}{36\gamma^{4}} \Big] \Big) \\ &\leq 2 \Big(\frac{\gamma^{2}}{\sqrt{n}} \Big)^{p} \Big(\sup_{y \ge 0} y^{\frac{p}{2}} \exp \Big[-\frac{y}{36} \Big] \Big) \Big(d^{2} \exp \Big[-\frac{n \zeta^{2}}{4\gamma^{4}} \Big] \Big) \le 2 \Big(\frac{\gamma^{2}}{\sqrt{n}} \Big)^{p} \Big(\sup_{y \ge 0} y^{\frac{p}{2}} \exp \Big[-\frac{y}{36} \Big] \Big) d^{2-4\delta}, \end{split}$$

where the second inequality is from Hoeffding's inequality (Proposition 3), while we plug in γ, ζ and use $C_5 \ge 4C_4^2$ in the last line. Some calculus show $\sup_{y\ge 0} y^{p/2} \exp(-y/36) \le D_2^p p^{p/2}$. Then we plug in the above inequality and the value of γ (2.15), for some D_3 we have

$$R_{21} \le d^{2-4\delta} \left(2D_2 \gamma^2 \sqrt{\frac{p}{n}} \right)^p \le d^{2-4\delta} \left(D_3 \sqrt{M} \left[\frac{\delta}{n \log d} \right]^{\frac{1}{4}} \right)^p$$

<u>Bound of R_{22} </u>. This is the same as the corresponding part in the proof of Theorem 3. In brief, we can show an upper bound of the same form, but with different value of γ^2 (given in (2.15)):

$$R_{22} \le d^{2-\frac{\delta}{2}} \left(D_{4,0} \gamma^2 \sqrt{\frac{p}{n}} \right)^p \le d^{2-\frac{\delta}{2}} \left(D_4 \sqrt{M} \left[\frac{\delta}{n \log d} \right]^{\frac{1}{4}} \right)^p$$

<u>Bound of R₂₃.</u> Note that $|\mathbb{E}\breve{\sigma}_{ij} - \sigma^*_{ij}| = |\mathbb{E}(\widetilde{X}_{k,i}\widetilde{X}_{k,j} - X_{k,i}X_{k,j})|$ is constant which has been bounded in the proof of Theorem 4. In particular, (A.13) gives $|\mathbb{E}\breve{\sigma}_{ij} - \sigma^*_{ij}| \le \frac{2}{C_3^2}\sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}$. By combining with $\mathbb{P}(\mathscr{A}_{ij}^c) \le 2d^{-\delta}$, we bound R_{23} via

$$R_{23} \le |\mathbb{E}\breve{\sigma}_{ij} - \sigma_{ij}^*|^p \sum_{i,j} 2d^{-\delta} \le d^{2-\delta} \Big(D_5 \sqrt{M} \Big[\frac{\delta \log d}{n} \Big]^{\frac{1}{4}} \Big)^p,$$

Now we are in a position to put things together. By combining the upper bounds for R_{2i} , i = 1, 2, 3, we have $R_{21} + R_{22} + R_{23} \leq D_6^p d^{2-\frac{\delta}{2}} M^{p/2} \left[\frac{\delta \log d}{n}\right]^{p/4}$. We further substitute it into (A.16), and recall $p = \frac{\delta}{4}$, $\delta \geq 4$, we obtain

$$R_2 \le d^{1-\frac{\delta}{4}} \left(6D_6 \sqrt{M} \left[\frac{\delta \log d}{n} \right]^{\frac{1}{4}} \right)^p \le \left(6D_6 \sqrt{M} \left[\frac{\delta \log d}{n} \right]^{\frac{1}{4}} \right)^p.$$

When $\delta \log d/n$ is small enough, this upper bound for R_2 is smaller than the obtained bound for R_1 . Thus, we know there exists absolute constant D_7 such that

$$\mathbb{E}\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\text{op}}^p \le \left(D_7 s M^{(1-q)/2} \left[\frac{\delta \log d}{n}\right]^{(1-q)/4}\right)^p,$$

(2.21) follows. We further use Markov inequality:

$$\mathbb{P}\left(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\mathrm{op}} \ge e^4 D_7 s M^{(1-q)/2} \left[\frac{\delta \log d}{n}\right]^{(1-q)/4}\right)$$
$$= \mathbb{P}\left(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*\|_{\mathrm{op}}^p \ge \left[e^4 D_7 s M^{(1-q)/2} \left[\frac{\delta \log d}{n}\right]^{(1-q)/4}\right]^p\right) \le \exp(-4p) = \exp(-\delta),$$

this displays (2.22) and concludes the proof.

B Proofs: Sparse Linear Regression

Proof of Lemma 2. The proof is obtained by modifying and combining Lemma 1 in [72] and Theorem 1 in [45].

I. From the detinition of Θ (3.3), we have

$$\mathcal{L}(\widehat{\Theta}) - \mathcal{L}(\Theta^*) \le \lambda \|\Theta^*\|_{\mathrm{nu}} - \lambda \|\widehat{\Theta}\|_{\mathrm{nu}}.$$
 (B.1)

By (3.4), some algebra delivers that

$$\mathcal{L}(\widehat{\boldsymbol{\Theta}}) - \mathcal{L}(\boldsymbol{\Theta}^*) = \frac{1}{2} \operatorname{vec}(\widehat{\boldsymbol{\Delta}})^T \boldsymbol{Q} \operatorname{vec}(\widehat{\boldsymbol{\Delta}}) - \left\langle \boldsymbol{B}, \widehat{\boldsymbol{\Delta}} \right\rangle + \operatorname{vec}(\boldsymbol{\Theta}^*)^T \boldsymbol{Q} \operatorname{vec}(\widehat{\boldsymbol{\Delta}}) = \frac{1}{2} \operatorname{vec}(\widehat{\boldsymbol{\Delta}})^T \boldsymbol{Q} \operatorname{vec}(\widehat{\boldsymbol{\Delta}}) + \left\langle \operatorname{mat}(\boldsymbol{Q} \cdot \operatorname{vec}(\boldsymbol{\Theta}^*)) - \boldsymbol{B}, \widehat{\boldsymbol{\Delta}} \right\rangle.$$
(B.2)

Since Q is positive semi-definite, combining with $\langle A_1, A_2 \rangle \leq ||A_1||_{\text{op}} ||A_2||_{\text{nu}}$, (3.5)

$$\mathcal{L}(\widehat{\Theta}) - \mathcal{L}(\Theta^*) \ge - \| \max(\boldsymbol{Q} \cdot \operatorname{vec}(\Theta^*)) - \boldsymbol{B} \|_{\operatorname{op}} \| \widehat{\Delta} \|_{\operatorname{nu}} \ge -\frac{\lambda}{2} \| \widehat{\Delta} \|_{\operatorname{nu}}.$$
(B.3)

II. Consider the SVD $\Theta^* = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$, where $U_1, V_1 \in \mathbb{R}^{d \times z}, U_2, V_2 \in \mathbb{R}^{d \times (d-z)}$ is a partition of singular vectors, $z \in \{0, 1, ..., d\}$ will be specified later. If $z \ge 1$ we consider two linear subspaces of $\mathbb{R}^{d \times d}$ defined as $\mathcal{M} = \{U_1 A_1 V_1^* : A_1 \in \mathbb{R}^{z \times z}\}$ and

$$\overline{\mathcal{M}} = \left\{ \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} : A_1 \in \mathbb{R}^{z \times z}, A_2 \in \mathbb{R}^{z \times (d-z)}, A_3 \in \mathbb{R}^{(d-z) \times z} \right\},$$

then let $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\overline{\mathcal{M}}}$ denote the projection onto \mathcal{M} and $\overline{\mathcal{M}}$ respectively. Given a matrix $\Delta \in \mathbb{R}^{d \times d}$, assume that $\Delta = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$, then $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\overline{\mathcal{M}}}$ have the explicit form

$$\mathcal{P}_{\mathcal{M}} \Delta = U_1 \Delta_{11} V_1^T \text{ and } \mathcal{P}_{\overline{\mathcal{M}}} \Delta = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

Besides, let $\mathcal{P}_{\mathcal{M}^{\perp}} \Delta = \Delta - \mathcal{P}_{\mathcal{M}} \Delta$, $\mathcal{P}_{\overline{\mathcal{M}}^{\perp}} \Delta = \Delta - \mathcal{P}_{\overline{\mathcal{M}}} \Delta$. Note that the nuclear norm is decomposable [72] with respect to the pair of subspaces $(\mathcal{M}, \overline{\mathcal{M}})$ since for any $\Delta_1, \Delta_2 \in \mathbb{R}^{d \times d}$, it holds that

$$\|\mathcal{P}_{\mathcal{M}}\Delta_{1} + \mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\Delta_{2}\|_{\mathrm{nu}} = \|\mathcal{P}_{\mathcal{M}}\Delta_{1}\|_{\mathrm{nu}} + \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\Delta_{2}\|_{\mathrm{nu}}.$$
 (B.4)

By using $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\overline{\mathcal{M}}}$, we have $\|\widehat{\Delta}\|_{nu} \leq \|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{nu} + \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\Delta}\|_{nu}$, plug in (B.3) and combine with (B.1), we obtain

$$\|\widehat{\boldsymbol{\Theta}}\|_{\mathrm{nu}} - \|\boldsymbol{\Theta}^*\|_{\mathrm{nu}} \le \frac{1}{2} \Big[\|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} + \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \Big].$$
(B.5)

In the special case z = 0, we just let $\mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\overline{\mathcal{M}}} = \mathbf{0}$, it can be easily verified that (B.4), (B.5) and what follow still hold.

III. In this part we derive (3.6). We calculate that

$$\begin{split} \|\widehat{\Theta}\|_{nu} - \|\Theta^*\|_{nu} &= \|\mathcal{P}_{\mathcal{M}}\Theta^* + \mathcal{P}_{\mathcal{M}^{\perp}}\Theta^* + \mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta} + \mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\Delta}\|_{nu} - \|\mathcal{P}_{\mathcal{M}}\Theta^* + \mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{nu} \\ &\geq \|\mathcal{P}_{\mathcal{M}}\Theta^*\|_{nu} + \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\Delta}\|_{nu} - \|\mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{nu} - \|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{nu} - \|\mathcal{P}_{\mathcal{M}}\Theta^*\|_{nu} - \|\mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{nu} \quad (B.6) \\ &= \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\Delta}\|_{nu} - 2\|\mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{nu} - \|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{nu}, \end{split}$$

note that we use decomposability (B.4) and triangle inequality in the third line. By combining (B.5), (B.6) we obtain $\|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\Delta}\|_{nu} \leq 3\|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{nu} + 4\|\mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{nu}$, it holds that

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \leq \|\mathcal{P}_{\overline{\mathcal{M}}^{\perp}}\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} + \|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} \leq 4(\|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\boldsymbol{\Delta}}\|_{\mathrm{nu}} + \|\mathcal{P}_{\mathcal{M}^{\perp}}\boldsymbol{\Theta}^{*}\|_{\mathrm{nu}}).$$
(B.7)

Assume the singular values of Θ^* are $\sigma_1(\Theta^*) \ge \dots \ge \sigma_d(\Theta^*)$. Instead of choosing z directly we choose a threshold $\tau > 0$ and then let $z = \max\{\{0\} \cup \{w \in [d] : \sigma_w(\Theta^*) \ge \tau\}\}$. Since $\operatorname{rank}(\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}) \le 2z$, we have $\|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{\operatorname{nu}} \le \sqrt{2z}\|\widehat{\Delta}\|_{\operatorname{F}}$. Moreover, by (3.2) we have

$$z\tau^q \leq \sum_{k=1}^z \sigma_k(\Theta^*)^q \leq \sum_{k=1}^d \sigma_k(\Theta^*)^q \leq r,$$

which implies $z \leq r\tau^{-q}$. Therefore, we have $\|\mathcal{P}_{\overline{\mathcal{M}}}\widehat{\Delta}\|_{\mathrm{nu}} \leq \sqrt{2r\tau^{-q/2}}\|\widehat{\Delta}\|_{\mathrm{F}}$. By simple algebra we can bound the last term in (B.7) by

$$\|\mathcal{P}_{\mathcal{M}^{\perp}}\Theta^*\|_{\mathrm{nu}} = \sum_{k=z+1}^d \sigma_k(\Theta^*) = \sum_{k=z+1}^d \sigma_k(\Theta^*)^q \sigma_k(\Theta^*)^{1-q} \le r\tau^{1-q}.$$

By putting pieces together, we obtain

$$\|\widehat{\mathbf{\Delta}}\|_{\mathrm{nu}} \le 4\left(\sqrt{2r\tau^{-\frac{q}{2}}}\|\widehat{\mathbf{\Delta}}\|_{\mathrm{F}} + r\tau^{1-q}\right), \ \forall \tau > 0.$$

We only consider $\widehat{\Delta} \neq 0$, then we choose $\tau = \left(\frac{\|\widehat{\Delta}\|_{\mathrm{F}}}{\sqrt{r}}\right)^{2/(2-q)}$, then we obtain (3.6). **IV.** Assume we have RSC (3.7), we derive the convergence rate. With RSC, from (B.2) we have tighter estimation than (B.3):

$$\mathcal{L}(\widehat{\Theta}) - \mathcal{L}(\Theta^*) \geq \frac{1}{2}\kappa \|\widehat{\Delta}\|_{\mathrm{F}}^2 - \frac{\lambda}{2}\|\widehat{\Delta}\|_{\mathrm{nu}}.$$

On the other hand we have $\mathcal{L}(\widehat{\Theta}) - \mathcal{L}(\Theta^*) \leq \lambda \|\widehat{\Delta}\|_{nu}$ from (B.1). By combining them we obtain $\|\widehat{\Delta}\|_{nu} \geq \frac{\kappa}{3\lambda} \|\widehat{\Delta}\|_{F}^{2}$. Then plug in (3.6), the bound for Frobenius norm in (3.8) follows. Again plug it into (3.6) we obtain the bound for nuclear norm.

Proof of Corollary 2. (3.9) can be recast as a trace regression $Y_k = \langle X_{k,tr}, \Theta_{tr}^* \rangle + \epsilon_k$, where $X_{k,tr} = \text{diag}(X_k)$, $\Theta_{tr}^* = \text{diag}(\Theta^*)$. Consider the convex set $S = \{\Theta \in \mathbb{R}^{d \times d} : \Theta = \text{diag}(\Theta), \|\Theta\|_{\max} \leq R\}$, let $B_{tr} = \text{diag}(B)$, and $Q_{tr} \in \mathbb{R}^{d^2 \times d^2}$ is the matrix whose submatrix constituted of the rows and columns with numbering in $\{1, d+2, 2d+3, ..., d^2\}$ is $Q \in \mathbb{R}^{d \times d}$, and the rows and columns not in $\{1, d+2, 2d+3, ..., d^2\}$ are all zero. Obviously, Q_{tr} is positive semi-definite. It is not hard to see that $\widehat{\Theta}$ defined by (3.11) is equivalent to finding the diagonal matrix $\widehat{\Theta}_{tr}$ via

$$\widehat{\boldsymbol{\Theta}}_{tr} \in \underset{\boldsymbol{\Theta} \in \mathcal{S}}{\operatorname{arg\,min}} \ \mathcal{L}(\boldsymbol{\Theta}) + \lambda \|\boldsymbol{\Theta}\|_{\operatorname{nu}},$$

where the loss function is given by $\mathcal{L}(\Theta) = \frac{1}{2} \operatorname{vec}(\Theta)^T Q_{tr} \operatorname{vec}(\Theta) - \langle B_{tr}, \Theta \rangle$, and then let $\widehat{\Theta}$ be the main diagonal of $\widehat{\Theta}_{tr}$. Then all the results follow by using Lemma 2.

B.1 Sub-Gaussian Data

Proof of Theorem 7. To use Corollary 2 we only need to establish (3.12), (3.14).

I. We first show that when $(\log n)^2 \log d/n$ is sufficiently small, Σ is positive definite with high probability. By Assumption 3 and Theorem 3 we have

$$\mathbb{P}\Big(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}}\|_{\text{op}} \le D_1 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big) \ge 1 - \exp(-\delta).$$
(B.8)

Under sufficiently small $(\log n)^2 \log d/n$ we have $\|\widehat{\Sigma} - \Sigma_{XX}\|_{op} \leq \kappa_0$ with probability higher than $1 - \exp(-\delta)$. Use $\lambda_{\min}(\cdot)$ to denote the smallest eigenvalue for a symmetric matrix. Combining with $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$ in Assumption 3, we obtain

$$\lambda_{\min}(\widehat{\Sigma}) \ge \lambda_{\min}(\Sigma_{XX}) - \|\widehat{\Sigma} - \Sigma_{XX}\|_{\text{op}} \ge \kappa_0, \tag{B.9}$$

which implies that $\hat{\Sigma}$ is positive definite, and (3.14) holds.

II. It remains to bound $\|\widehat{\Sigma}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$ and show (3.12) holds with high probability. Let $\Sigma_{YX} = \mathbb{E}Y_k X_k$ and first note that

$$\Sigma_{YX} = \mathbb{E}(Y_k X_k) = \mathbb{E}(X_k X_k^T \Theta^* + \epsilon_k X_k) = \mathbb{E}(X_k X_k^T) \Theta^* = \Sigma_{XX} \Theta^*.$$

By repeating the proof of Theorem 1, we have the element-wise error for $\widehat{\Sigma}_{YX}$

$$\mathbb{P}\Big(\|\widehat{\Sigma}_{YX} - \Sigma_{YX}\|_{\max} \le D_2 \sigma^2 \log n \sqrt{\frac{\delta \log d}{n}}\Big) \ge 1 - 2d^{1-\delta}.$$
(B.10)

We now combine (B.8) and (B.10), it holds with probability higher than $1 - 2d^{1-\delta} - \exp(-\delta)$ that

$$\|\widehat{\boldsymbol{\Sigma}}\Theta^* - \widehat{\boldsymbol{\Sigma}}_{YX}\|_{\max} \le \|\widehat{\boldsymbol{\Sigma}}\Theta^* - \boldsymbol{\Sigma}_{XX}\Theta^*\|_{\max} + \|\boldsymbol{\Sigma}_{YX} - \widehat{\boldsymbol{\Sigma}}_{YX}\|_{\max}$$
$$\le \|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_{XX}\|_{\text{op}}\|\Theta^*\|_2 + \|\boldsymbol{\Sigma}_{YX} - \widehat{\boldsymbol{\Sigma}}_{YX}\|_{\max} \le (D_1R + D_2)\sigma^2\log n\sqrt{\frac{\delta\log d}{n}}.$$
(B.11)

Thus, we can choose sufficiently large C_6 in (3.19) such that $C_6 \ge 2(D_1R + D_2)$, then (3.12) holds with high probability. Now that (3.12) and (3.14) have been verified, Corollary 2 gives (3.15). We further substitute (3.19) into (3.15) and conclude the proof.

B.2 Heavy-tailed Data

Proof of Theorem 8. The proof is parallel to Theorem 7. By Assumption 3 and Theorem 6, we have the probability tail for operator norm deviation

$$\mathbb{P}\Big(\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_{XX}\|_{\text{op}} \le D_1 \sqrt{M} \Big[\frac{\delta \log d}{n}\Big]^{1/4}\Big) \ge 1 - \exp(-\delta), \tag{B.12}$$

when $\log d/n$ is sufficiently small, we can assume $\|\widehat{\Sigma} - \Sigma_{XX}\|_{op} \leq \kappa_0$ with probability higher than $1 - \exp(-\delta)$. This, together with $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$ given in Assumption 3, gives $\lambda_{\min}(\widehat{\Sigma}) \geq \kappa_0$ under the same probability. Thus, with high probability $\widehat{\Sigma}$ is positive definite and (3.14) holds.

It remains to establish (3.12) and apply Corollary 2. By repeating the proof of Theorem 4, we can show the max-norm error for $\widehat{\Sigma}_{YX}$ to approximate $\Sigma_{YX} = \mathbb{E}Y_k X_k$ as

$$\mathbb{P}\left(\|\widehat{\Sigma}_{YX} - \Sigma_{YX}\|_{\max} \le D_2 \sqrt{M} \left[\frac{\delta \log d}{n}\right]^{1/4}\right) \ge 1 - 2d^{1-\delta}.$$
(B.13)

Now we combine (B.12) and (B.13), with probability higher than $1 - \exp(-\delta) - 2d^{1-\delta}$ it yields

$$\|\widehat{\Sigma}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max} \le \|\widehat{\Sigma}\Theta^* - \Sigma_{XX}\Theta^*\|_{\max} + \|\Sigma_{YX} - \widehat{\Sigma}_{YX}\|_{\max}$$

$$\le \|\widehat{\Sigma} - \Sigma_{XX}\|_{\mathrm{op}}\|\Theta^*\|_2 + \|\Sigma_{YX} - \widehat{\Sigma}_{YX}\|_{\max} \le (D_1R + D_2)\sqrt{M} \Big[\frac{\delta \log d}{n}\Big]^{1/4}.$$
(B.14)

Thus, in (3.22) we can choose sufficiently large C_7 such that $C_7 \ge 2(D_2R + D_1)$, then we verify $\lambda \ge 2 \|\widehat{\Sigma}\Theta^* - \widehat{\Sigma}_{YX}\|_{\text{max}}$. Now we can use (3.15) in Corollary 2 and plug in (3.22), the desired error bounds follow.

B.3 1-bit Compressed Sensing

Proof of Theorem 9. We prove the error bound based on Corollary 2. Evidently, we need to show setting $\lambda = C_8 \sqrt{\frac{\delta \log d \log n}{n}}$ with sufficiently large C_8 can guarantee $\lambda \geq 2 \|\widehat{\Sigma}_{XX}\Theta^* - \widehat{\Sigma}_{YX}\|_{\text{max}}$. First we use triangle inequality to obtain

$$\begin{split} \|\widehat{\Sigma}_{XX}\Theta^{*} - \widehat{\Sigma}_{YX}\|_{\max} &= \left\|\frac{1}{n}\sum_{k=1}^{n}X_{k}X_{k}^{T}\Theta^{*} - \frac{1}{n}\sum_{k=1}^{n}\gamma\cdot\dot{Y}_{k}X_{k}\right\|_{\max} \\ &= \left\|\frac{1}{n}\sum_{k=1}^{n}(Y_{k} - \gamma\cdot\dot{Y}_{k})X_{k} - \frac{1}{n}\sum_{k=1}^{n}\epsilon_{k}X_{k}\right\|_{\max} \\ &\leq \left\|\frac{1}{n}\sum_{k=1}^{n}(Y_{k} - \gamma\cdot\dot{Y}_{k})X_{k} - \mathbb{E}\left[(Y_{k} - \gamma\cdot\dot{Y}_{k})X_{k}\right]\right\|_{\max} + \left\|\mathbb{E}\left[(Y_{k} - \gamma\cdot\dot{Y}_{k})X_{k}\right]\right\|_{\max} \\ &+ \left\|\frac{1}{n}\sum_{k=1}^{n}\epsilon_{k}X_{k}\right\|_{\max} := R_{1} + R_{2} + R_{3}, \end{split}$$
(B.15)

where we use $Y_k = X_k^T \Theta^* + \epsilon_k$ in the second line.

Bound of R_1 . Note that $||X_k||_{\psi_2} \leq \sigma_1$, $||\epsilon_k||_{\psi_2} \leq \sigma_2$, $||\Theta^*|| \leq R$ imply $||Y_k||_{\psi_2} = O(R\sigma_1 + \sigma_2) = O(1)$, (since we assume σ_1, σ_2, R are absolute constants). Thus, we have $||Y_k - \gamma \cdot \dot{Y}_k||_{\psi_2} = O(1 + \gamma)$. Denote the *j*-th entry of X_k by $X_{k,j}$, then $X_{k,j} \leq \sigma_1$. Hence, by (1.2) we obtain $||(Y_k - \gamma \cdot \dot{Y}_k)X_{k,j}||_{\psi_1} \leq ||Y_k - \gamma \cdot \dot{Y}_k||_{\psi_2} ||X_{k,j}||_{\psi_2} = O(1 + \gamma)$. Therefore, by Bernstein's inequality in Proposition 4 we obtain

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}(Y_{k}-\gamma\cdot\dot{Y}_{k})X_{k,j}-\mathbb{E}\left((Y_{k}-\gamma\cdot\dot{Y}_{k})X_{k,j}\right)\right|\geq t\right) \\
\leq 2\exp\left(-D_{1}n\min\left\{\frac{t^{2}}{(1+\gamma)^{2}},\frac{t}{1+\gamma}\right\}\right), \ \forall t>0.$$
(B.16)

Moreover, we use $\gamma = C'_8 \sqrt{\log n}$ and a union bound to obtain

$$\mathbb{P}(R_1 \ge t) \le 2d \exp\left(-D_2 n \min\left\{\frac{t^2}{\log n}, \frac{t}{\sqrt{\log n}}\right\}\right).$$

Thus, setting $t = D_3 \sqrt{\frac{\delta \log n \log d}{n}}$ with large D_3 yields $R_1 \leq D_3 \sqrt{\frac{\delta \log n \log d}{n}}$ with probability at least $1 - 2d^{1-\delta}$.

<u>Bound of R₂.</u> By Lemma 1 when $|Y_k| \leq \gamma$ we have $\mathbb{E}_{\Lambda_k}(\gamma \cdot \dot{Y}_k) = Y_k$. Use this fact and Cauchy-Schwarz inequality, we can first bound R_2 from above as

$$R_{2} = \left\| \mathbb{E} \left(Y_{k} - \gamma \cdot \dot{Y}_{k} \right) X_{k} \right\|_{\max} = \left\| \mathbb{E} \left(Y_{k} - \gamma \cdot \dot{Y}_{k} \right) X_{k} \mathbb{1} \left(|Y_{k}| > \gamma \right) \right\|_{\max}$$

$$\leq \max_{j \in [d]} \mathbb{E} \left(|Y_{k} X_{k,j}| \mathbb{1} \left(|Y_{k}| > \gamma \right) \right) \leq \max_{j \in [d]} \sqrt{\mathbb{E} \left[|Y_{k}|^{2} |X_{k,j}|^{2} \right]} \sqrt{\mathbb{P} \left(|Y_{k}| > \gamma \right)}$$

$$\leq \max_{j \in [d]} \sqrt{\frac{1}{2} \left(\mathbb{E} |Y_{k}|^{4} + \mathbb{E} |X_{k,j}|^{4} \right) \mathbb{P} \left(|Y_{k}| > \gamma \right)} \lesssim \exp(-D_{4} \gamma^{2}) = n^{-n^{4} (C_{8}')^{2}},$$

where the last inequality follows from $||Y_k||_{\psi_2}$, $||X_{k,j}||_{\psi_2} = O(1)$, Proposition 1 and the choice $\gamma = C'_8 \sqrt{\log n}$. Thus, as long as C'_8 is sufficiently large, we have $R_2 = O(\sqrt{\frac{\delta \log n \log d}{n}})$.

<u>Bound of R3.</u> For $j \in [d]$, by (1.2) it is evident that $\|\epsilon_k X_{k,j}\|_{\psi_1} \leq \|\epsilon_k\|_{\psi_2} \|X_{k,j}\|_{\psi_2} \leq \sigma_1 \sigma_2 = O(1)$. Thus, Bernstein's inequality (Proposition 4) followed by a union bound gives

$$\mathbb{P}(R_3 \ge t) \le 2d \exp\left(-D_{4,1}n\min\{t^2,t\}\right), \ \forall \ t > 0.$$
(B.17)

We further set $t = \sqrt{\frac{\delta \log d}{D_{4,1}n}}$ and obtain

$$\mathbb{P}\Big(R_3 \le \sqrt{\frac{\delta \log d}{D_{4,1}n}}\Big) \ge 1 - 2d^{1-\delta}.$$

By (2.6) we can assume $\sigma < \gamma \lesssim \sigma \sqrt{\log n}$. Thus, by (B.15) and the upper bounds for R_1, R_2, R_3 , with probability higher than $1 - 4d^{2-\delta}$ we have

$$\|\widehat{\boldsymbol{\Sigma}}_{XX}\Theta^* - \widehat{\boldsymbol{\Sigma}}_{YX}\|_{\max} \lesssim \sqrt{\frac{\delta \log n \log d}{n}}$$

Therefore, we can choose sufficiently large C_8 in (3.25) to guarantee $\lambda \geq 2 \|\widehat{\Sigma}_{XX}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$ holds with high probability. By Corollary 2, it already leads to (3.13), a relation that facilitates the following discussions.

Now that (3.12) has been verified, we turn to consider the RSC (3.14). When $\frac{\delta \log d}{n}$ is sufficiently small, combining with $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$, Lemma 2(a) in [45] gives

$$\mathbb{P}\left(\widehat{\Delta}^T \widehat{\Sigma}_{XX} \widehat{\Delta} \ge \kappa_0 \|\widehat{\Delta}\|_2^2 - \frac{D_5 \delta \log d}{n} \|\widehat{\Delta}\|_1^2\right) \ge 1 - 3d^{1-\delta}.$$

This event, together with (3.13), implies

$$\widehat{\Delta}^T \widehat{\Sigma}_{XX} \widehat{\Delta} \ge \kappa_0 \|\widehat{\Delta}\|_2^2 - D_6 \cdot \frac{\delta \log d}{n} \cdot s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}}.$$
(B.18)

We proceed the proof upon the condition (B.18) and divide it into the following two cases.

<u>Case 1.</u> If $D_6 \cdot \frac{\delta \log d}{n} \cdot s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}} \leq \frac{\kappa_0}{2} \|\widehat{\Delta}\|_2^2$, (B.18) gives the RSC (3.14) with $\kappa = \frac{\kappa_0}{2}$. Thus, we can invoke (3.15) in Corollary 2 and then plug in the value of λ in (3.25). This displays the desired error bounds.

<u>Case 2.</u> Otherwise, it holds that

$$D_6 \cdot \frac{\delta \log d}{n} \cdot s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}} \ge \frac{\kappa_0}{2} \|\widehat{\Delta}\|_2^2. \tag{B.19}$$

With no loss of generality, we assume $\widehat{\Delta} \neq 0$. Under the scaling that $\sqrt{s} \left(\sqrt{\frac{\delta \log d}{n}} \right)^{1-q/2}$ is sufficiently small we have $q \in (0,1)$ (Since when q = 0, $D_6 \cdot \frac{\delta \log d}{n} \cdot s^{\frac{2}{2-q}} < \frac{\kappa_0}{2}$ together with (B.19) gives $\widehat{\Delta} = 0$). Again use sufficiently small $\sqrt{s} \left(\sqrt{\frac{\delta \log d}{n}} \right)^{1-q/2}$, (B.19) delivers

$$\|\widehat{\Delta}\|_2 \lesssim \left[\sqrt{s}\left(\sqrt{\frac{\delta \log d}{n}}\right)^{1-\frac{q}{2}}\right]^{\frac{2}{q}} \le \sqrt{s}\left(\sqrt{\frac{\delta \log d}{n}}\right)^{1-\frac{q}{2}}.$$

This, together with (3.13), gives the upper bound for $\|\widehat{\Delta}\|_1$ as

$$\|\widehat{\Delta}\|_1 \lesssim s\left(\sqrt{\frac{\delta \log d}{n}}\right)^{1-q}.$$

Thus, we conclude the proof.

Proof of Theorem 10. The proof is still based on Corollary 2. First let us verify the crucial relation $\lambda \geq 2 \|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$. Note that $\Sigma_{XX}\Theta^* = \mathbb{E}Y_kX_k$, by triangle inequality we can divide it into three terms R_i , $1 \leq i \leq 3$

$$\begin{aligned} \|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max} &\leq \left\| \left(\widehat{\Sigma}_{\tilde{X}\tilde{X}} - \Sigma_{XX} \right)\Theta^* \right\|_{\max} + \left\| \mathbb{E} \left(\gamma \cdot \dot{Y}_k \widetilde{X}_k - Y_k X_k \right) \right\|_{\max} \\ &+ \left\| \frac{1}{n} \sum_{k=1}^n \gamma \cdot \dot{Y}_k \widetilde{X}_k - \mathbb{E} \left(\gamma \cdot \dot{Y}_k \widetilde{X}_k \right) \right\|_{\max} := R_1 + R_2 + R_3. \end{aligned}$$
(B.20)

<u>Bound of R_1 </u>. We first decompose $\|\widehat{\Sigma}_{\tilde{X}\tilde{X}} - \Sigma_{XX}\|_{\max}$ as

$$\left\|\widehat{\boldsymbol{\Sigma}}_{\tilde{X}\tilde{X}} - \boldsymbol{\Sigma}_{XX}\right\|_{\max} \le \left\|\widehat{\boldsymbol{\Sigma}}_{\tilde{X}\tilde{X}} - \mathbb{E}\widetilde{X}_{k}\widetilde{X}_{k}^{T}\right\|_{\max} + \left\|\mathbb{E}\left(X_{k}X_{k}^{T} - \widetilde{X}_{k}\widetilde{X}_{k}^{T}\right)\right\|_{\max} := R_{11} + R_{12}$$

Let us deal with them element-wisely. For R_{11} and any $(i, j) \in [d] \times [d]$, recall that the truncated covariate satisfies $|\tilde{X}_{k,i}| \leq \eta_X$, combining with (3.21) it gives

$$\begin{cases} \sum_{k=1}^{n} \mathbb{E} \left(\widetilde{X}_{k,i} \widetilde{X}_{k,j} \right)^{2} \leq \sum_{k=1}^{n} \mathbb{E} X_{k,i}^{2} X_{k,j}^{2} \leq \sum_{k=1}^{n} \frac{1}{2} \left(\mathbb{E} X_{k,i}^{4} + \mathbb{E} X_{k,j}^{4} \right) \leq nM \\ \sum_{k=1}^{n} \mathbb{E} \left(\widetilde{X}_{k,i} \widetilde{X}_{k,j} \right)_{+}^{q} \leq \sum_{k=1}^{n} \mathbb{E} \left| \widetilde{X}_{k,i} \widetilde{X}_{k,j} \right|^{q} \leq (\eta_{X}^{2})^{q-2} \sum_{k=1}^{n} \mathbb{E} \left(\widetilde{X}_{k,i} \widetilde{X}_{k,j} \right)^{2} \leq nM \cdot (\eta_{X}^{2})^{q-2}, \forall q \geq 3 \end{cases}$$

Thus, by the version of Bernstein's inequality given in Theorem 2.10 in [14], we obtain

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n}\widetilde{X}_{k,i}\widetilde{X}_{k,j} - \mathbb{E}\widetilde{X}_{k,i}\widetilde{X}_{k,j}\Big| > \sqrt{\frac{2Mt}{n}} + \frac{\eta_X^2 t}{n}\Big) \le \exp(-t), \ \forall \ t > 0$$

Moreover, we can use an union bound and get

$$\mathbb{P}\left(R_{11} > \sqrt{\frac{2Mt}{n}} + \frac{\eta_X^2 t}{n}\right) \le d^2 \cdot \exp(-t), \ \forall \ t > 0.$$

Thus, we set $t = \delta \log d$ and plug in $\eta_X \simeq \left(\frac{n}{\log d}\right)^{1/4}$, then with probability at least $1 - 2d^{2-\delta}$ we have $R_{11} \lesssim \sqrt{\frac{\delta \log d}{n}}$. We now turn to R_{12} and have the (i, j)-th entry bounded by

$$\left|\mathbb{E}\left(X_{k,i}X_{k,i}-\widetilde{X}_{k,i}\widetilde{X}_{k,j}\right)\right| \leq \mathbb{E}|X_{k,i}X_{k,j}| \left(\mathbb{1}\left(|X_{k,i}| > \eta_X\right) + \mathbb{1}\left(|X_{k,j}| > \eta_X\right)\right).$$

The two terms can be bounded likewise, so we only deal with one of them by Cauchy-Schwarz inequality and (3.21):

$$\mathbb{E}|X_{k,i}X_{k,j}|\mathbb{1}(|X_{k,i}| > \eta_X) \le \sqrt{\mathbb{E}|X_{k,i}X_{k,j}|^2}\sqrt{\mathbb{P}(|X_{k,i}| > \eta_X)}$$
$$\le \sqrt{\frac{1}{2}(\mathbb{E}|X_{k,i}|^4 + \mathbb{E}|X_{k,j}|^4)}\sqrt{\frac{\mathbb{E}|X_{k,i}|^4}{\eta_X^4}} \le \frac{M}{\eta_X^2} \lesssim \sqrt{\frac{\delta \log d}{n}}.$$

Therefore, with high probability we have

$$R_1 \le \|\widehat{\boldsymbol{\Sigma}}_{\tilde{X}\tilde{X}} - \boldsymbol{\Sigma}_{XX}\|_{\max} \|\Theta^*\|_1 \lesssim (R_{11} + R_{12}) \lesssim \sqrt{\frac{\delta \log d}{n}}.$$

Bound of R_2 . We consider the *j*-th entry. Note that $\gamma > \eta_Y$, Lemma 1 gives

$$|\mathbb{E}(\gamma \cdot \dot{Y}_k \widetilde{X}_{k,j} - Y_k X_{k,j})| = |\mathbb{E}(\widetilde{Y}_k \widetilde{X}_{k,j} - Y_k X_{k,j})| \le \mathbb{E}(|Y_k X_{k,j}| \mathbb{1}(|Y_k| > \eta_Y) + \mathbb{1}(|X_{k,j}| > \eta_X))$$

By Cauchy-Schwarz inequality, (3.21) and the value of η_Y , we obtain

$$\mathbb{E}|Y_k X_{k,j}|\mathbb{1}(|Y_k| > \eta_Y) \le \sqrt{\mathbb{E}|Y_k X_{k,j}|^2 \cdot \mathbb{P}(|Y_k| > \eta_Y)} \le \frac{M}{\eta_Y^2} \lesssim \left(\frac{\delta \log d}{n}\right)^{\frac{1}{3}}$$

Similarly, it holds that $\mathbb{E}|Y_k X_{k,j}| \mathbb{1}(|X_{k,j} > \eta_X) \leq \frac{M}{\eta_X^2} \lesssim \sqrt{\frac{\delta \log d}{n}}$. Since this is valid for any $j \in [d]$, we obtain $R_2 \lesssim \left(\frac{\delta \log d}{n}\right)^{1/3}$.

Bound of R_3 . We consider the *j*-th entry first. Recall that $|\widetilde{X}_{k,j}| \leq \eta_X$, and by (3.21) we know $\mathbb{E}\widetilde{X}_{k,j}^2 \leq \mathbb{E}|X_{k,j}|^2 \leq \sqrt{M}$, thus we have

$$\begin{cases} \sum_{k=1}^{n} \mathbb{E}\left(\gamma \cdot \dot{Y}_{k} \widetilde{X}_{k,j}\right)^{2} = \gamma^{2} \sum_{k=1}^{n} \mathbb{E} \widetilde{X}_{k,j}^{2} \le n\sqrt{M}\gamma^{2} \\ \sum_{k=1}^{n} \mathbb{E}\left(\gamma \cdot \dot{Y}_{k} \widetilde{X}_{k,j}\right)_{+}^{q} \le \gamma^{q} \sum_{k=1}^{n} \mathbb{E}|\widetilde{X}_{k,j}|^{q} \le n\sqrt{M}\gamma^{2}(\gamma \cdot \eta_{X})^{q-2}, \ \forall \ q \ge 3. \end{cases}$$
(B.21)

Now, we can invoke the Bernstein's inequality given in Theorem 2.10 in [14] and obtain

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n}\gamma\cdot\dot{Y}_{k}\widetilde{X}_{k,j}-\mathbb{E}\gamma\cdot\dot{Y}_{k}\widetilde{X}_{k,j}\Big|>\gamma\sqrt{\frac{2\sqrt{Mt}}{n}}+\frac{\gamma\cdot\eta_{X}t}{n}\Big)\leq\exp(-t),\;\forall\;t>0.$$

Thus, for some absolute constant hidden behind " \gtrsim ", a union bound gives

$$\mathbb{P}\left(R_3 \gtrsim \gamma \sqrt{\frac{t}{n}} + \frac{\gamma \cdot \eta_X \cdot t}{n}\right) \le d \cdot \exp(-t), \ \forall \ t > 0.$$
(B.22)

We set $t = \delta \log d$ and plug in our choices $\eta_X \approx \left(\frac{n}{\delta \log d}\right)^{1/4}$ and $\gamma \approx \left(\frac{n}{\delta \log d}\right)^{1/6}$, it yields that $R_3 \lesssim \left(\frac{\delta \log d}{n}\right)^{1/3}$ holds with probability at least $1 - d^{1-\delta}$. Now combining the upper bounds for $R_i, 1 \leq i \leq 3$ and (B.20), we can choose $\lambda = 1 + \frac{1}{2}$

Now combining the upper bounds for $R_i, 1 \leq i \leq 3$ and (B.20), we can choose $\lambda = C_{12} \left(\frac{\delta \log d}{n}\right)^{1/3}$ with sufficiently large C_{12} to guarantee $\lambda \geq 2 \|\widehat{\Sigma}_{\tilde{X}\tilde{X}}\Theta^* - \widehat{\Sigma}_{YX}\|_{\max}$. Note that $\sum_{XX}\Theta^* = \mathbb{E}Y_k X_k$. By Corollary 2 under the same probability we have (3.13), i.e., $\|\widehat{\Delta}\|_1 \leq 10s^{\frac{1}{2-q}} \|\widehat{\Delta}\|_2^{\frac{2-2q}{2-q}}$.

To invoke Corollary 2 we still need to establish the RSC (3.14). Note that our choice of the truncation parameter η_X is the same as [45], so we can use Lemma 2(b) therein⁴. Combining with $\lambda_{\min}(\Sigma_{XX}) \geq 2\kappa_0$ and (3.13), it gives

$$\mathbb{P}\left(\widehat{\Delta}^T \widehat{\Sigma}_{\tilde{X}\tilde{X}} \widehat{\Delta} \ge 2\kappa_0 \|\widehat{\Delta}\|_2^2 - D_1 \sqrt{\frac{\delta \log d}{n}} s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}}\right) \ge 1 - d^{2-\sqrt{\delta}}.$$

⁴This result is presented with the probability term reversed in different versions of [45], but the proof

We assume the above event holds, and divide the discussion into two cases.

<u>Cases 1.</u> If $D_1 \sqrt{\frac{\delta \log d}{n}} s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}} \leq \kappa_0 \|\widehat{\Delta}\|_2^2$, we have $\widehat{\Delta}^T \widehat{\Sigma}_{\tilde{X}\tilde{X}} \widehat{\Delta} \geq \kappa_0 \|\widehat{\Delta}\|_2^2$, thus confirming the RSC (3.14). Therefore, we can use (3.15) in Corollary 2 and plug in $\lambda \asymp \left(\frac{\delta \log d}{n}\right)^{1/3}$ to yield the error bound $\|\widehat{\Delta}\|_2 \lesssim \sqrt{s} \left(\frac{\delta \log d}{n}\right)^{(1-\frac{q}{2})/3} := B_1.$ <u>Cases 2.</u> Otherwise, we assume

$$D_1 \sqrt{\frac{\delta \log d}{n}} s^{\frac{2}{2-q}} \|\widehat{\Delta}\|_2^{\frac{4-4q}{2-q}} > \kappa_0 \|\widehat{\Delta}\|_2^2. \tag{B.23}$$

With no loss of generality we assume $\widehat{\Delta} \neq 0$. If q = 0, under the scaling that $s\left(\sqrt{\frac{\delta \log d}{n}}\right)^{1-q/2}$ is sufficiently small, (B.23) can imply $\widehat{\Delta} = 0$. Thus, we assume $q \in (0, 1)$ without losing generality, then (B.23) gives

$$\|\widehat{\Delta}\|_2 \lesssim \left[s\left(\sqrt{\frac{\delta \log d}{n}}\right)^{1-q/2}\right]^{1/q} = s^{\frac{1}{q}} \left(\frac{\delta \log d}{n}\right)^{\frac{1}{2q}-\frac{1}{4}} := B_2.$$

Therefore, we obtain $\|\widehat{\Delta}\|_2 \lesssim \max\{B_1, B_2\} = B_1 \max\{1, \frac{B_2}{B_1}\}$. Because we have assumed the additional $s\left(\frac{\delta \log d}{n}\right)^{\frac{1}{2}-\frac{q}{3}} = O(1)$ for $q \in (0, 1)$, it leads to

$$\frac{B_2}{B_1} = \left(s\left(\frac{\delta \log d}{n}\right)^{\frac{1}{2} - \frac{q}{3}}\right)^{\frac{1}{q} - \frac{1}{2}} = O(1).$$

Therefore, we arrive at the desired upper bound $\|\widehat{\Delta}\|_2 \lesssim \sqrt{s} \left(\frac{\delta \log d}{n}\right)^{(1-q/2)/3}$. Combining with (3.13), the bound for $\|\widehat{\Delta}\|_1$ follows.

Proof of Theorem 11. We simply write $\mathscr{K}(s, R)$ as \mathscr{K} in this proof, and we use the shorthand Σ_s to denote the set of s-sparse vectors in \mathbb{R}^d .

We first use the sparse Varshamov-Gilbert (e.g., [82, Lemma 4.14]) to construct a packing set $\mathscr{K}_0 := \{\Theta_0^{(1)}, ..., \Theta_0^{(N)}\} \subset \Sigma_s$ such that

- for any $i \in [N]$, $\Theta_0^{(i)}$ has s non-zero entries that equal 1;
- $\log N \ge \frac{s}{8} \log \frac{d}{2s};$
- for $i \neq j$, $\Theta_0^{(i)}$ and $\Theta_0^{(j)}$ contain at least $\frac{s}{2}$ different entries.

Then we let $\hat{\alpha} = D_1 \gamma \sqrt{\frac{\log \frac{d}{2s}}{nK_u}}$ for some constant D_1 , and recall that we choose $\gamma \asymp \left(\frac{n}{\log d}\right)^{1/6}$. We consider the set of parameters $\mathscr{K}_1 = \hat{\alpha} \mathscr{K}_0 = \{\Theta^{(i)} = \hat{\alpha} \Theta_0^{(i)} : i \in [N]\}$. Because we assume $n \gtrsim K_u^{-1} \left(\frac{s}{R}\right)^3 \log \frac{d}{2s}$, we have $\|\Theta^{(i)}\|_1 = s\hat{\alpha} \leq R$, and so $\mathscr{K}_1 \subset \mathscr{K}$.

For $a \in \mathbb{R}$ we let $\mathsf{T}_{\eta}(a) = \operatorname{sign}(a) \min\{|a|, \eta\}$. It suffices to consider the noiseless case $\epsilon_k = 0$, and hence for underlying matrix Θ we have the observations

$$\mathscr{P}(\Theta) = \{ \dot{Y}_k = \operatorname{sign}(\mathsf{T}_{\eta_Y}(X_k^T\Theta) + \Lambda_k), k = 1, 2, ..., n \}.$$

therein is find and can yield what we need here.

Note that for any $i \neq j$

$$\frac{s}{2}\hat{\alpha}^2 \le \|\Theta^{(i)} - \Theta^{(j)}\|_2^2 \le 2s\hat{\alpha}^2.$$

By reduction to hypothesis testing and Fano's inequality (e.g., [82, Section 4]), we have

$$\inf_{\widehat{\Theta}} \sup_{\Theta \in \mathscr{K}} \mathbb{P}_{\Theta} \Big(\|\widehat{\Theta} - \Theta\|_2^2 > \frac{s}{8} \hat{\alpha}^2 \Big) \ge 1 - \frac{\frac{1}{N^2} \sum_{i,j=1}^N \mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)})) + \log 2}{\log N}.$$
(B.24)

Then we estimate $\mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)}))$. Let $\Theta^{(i)}$ be the underlying parameter, then because $\eta_Y < \frac{8}{9}\gamma$, the corresponding Y_k follows a (symmetrized) Bernoulli distribution with success probability

$$\mathbb{P}(\dot{Y}_k = 1) = \mathbb{P}\left(\Lambda_k > -\mathsf{T}_{\eta_Y}(X_k^T\Theta)\right) = \frac{\gamma + \mathsf{T}_{\eta_Y}(X_k^T\Theta)}{2\gamma} \in \left[\frac{1}{18}, \frac{17}{18}\right].$$

We use $\mathsf{KL}(p,q)$ to denote the KL divergence between Bernoulli distribution with success probability p and q, then [34, Lemma A.4] provides $\mathsf{KL}(p,q) \leq \frac{(p-q)^2}{q(1-q)}$. Because $\{\dot{Y}_k : k = 1, ..., n\}$ are independent, we have

$$\begin{split} \mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)})) &= \sum_{k=1}^{n} \mathsf{KL}\Big(\frac{1}{2} + \frac{\mathsf{T}_{\eta_{Y}}(X_{k}^{T}\Theta^{(i)})}{2\gamma}, \frac{1}{2} + \frac{\mathsf{T}_{\eta_{Y}}(X_{k}^{T}\Theta^{(j)})}{2\gamma}\Big) \\ &\leq \sum_{k=1}^{n} \frac{D_{2}}{\gamma^{2}} \Big(\mathsf{T}_{\eta_{Y}}(X_{k}^{T}\Theta^{(i)}) - \mathsf{T}_{\eta_{Y}}(X_{k}^{T}\Theta^{(j)})\Big)^{2} \leq \frac{D_{2}}{\gamma^{2}} \sum_{k=1}^{n} \Big(X_{k}^{T}(\Theta^{(i)} - \Theta^{(j)})\Big)^{2} \\ &\leq \frac{D_{2}K_{u}n}{\gamma^{2}} \big\|\Theta^{(i)} - \Theta^{(j)}\big\|_{2}^{2} \leq 2D_{2}s \frac{K_{u}n\hat{\alpha}^{2}}{\gamma^{2}} = 2D_{2}D_{1}^{2}s\log\frac{d}{2s}. \end{split}$$

Note that in the first inequality we use $\mathsf{KL}(p,q) \leq \frac{(p-q)^2}{q(1-q)}$, and because q is bounded away from 0 and 1, $\frac{1}{q(1-q)}$ is bounded by absolute constant, then in the following inequalities we use the assumption $\sum_{k=1}^{n} |X_k^T V|^2 \leq nK_u ||V||_2^2$ for $V \in \Sigma_{2s}$, finally we plug in our choice of $\hat{\alpha}$. Since the estimate is valid for any i, j, and $\log N \geq \frac{s}{8} \log \frac{d}{2s}$, so we can set D_1 sufficiently small so that under relatively large s, the right hand side of (B.24) is greater than $\frac{3}{4}$. Also, we perform some algebra to arrive at

$$\sqrt{s\hat{\alpha}} \asymp \left(\frac{\log \frac{d}{2s}}{\log d}\right)^{\frac{1}{6}} \sqrt{\frac{s}{K_u}} \left(\frac{\log \frac{d}{2s}}{n}\right)^{1/3}$$

Putting this into (B.24) completes the proof.

C Proofs: Low-rank Matrix Completion

C.1 Sub-Gaussian Data

Proof of Lemma 3. I. We first prove several facts that would be frequently used later. <u>Fact 1:</u> $\mathbb{E}X_k^T X_k = \mathbb{E}X_k X_k^T = I_d/d.$

Since X_k and X_k^T follow the same distribution, we only calculate $\mathbb{E} X_k^T X_k$. Equivalent to (4.2)

we can assume $\mathbf{X}_{\mathbf{k}} = e_{k(i)} e_{k(j)}^T$ where $(k(i), k(j)) \sim \operatorname{uni}([d] \times [d])$. Then we calculate that

$$\mathbb{E} \boldsymbol{X}_{\boldsymbol{k}}^{T} \boldsymbol{X}_{\boldsymbol{k}} = \mathbb{E}_{k(i),k(j)} e_{k(j)} e_{k(j)}^{T} e_{k(i)} e_{k(j)}^{T} = \mathbb{E}_{k(j)} e_{k(j)} e_{k(j)}^{T} e_{k(j)}^{T}$$
$$= \sum_{k(j)=1}^{d} d^{-1} e_{k(j)} e_{k(j)}^{T} = \boldsymbol{I}_{\boldsymbol{d}}/d.$$

<u>Fact 2</u>: Given random matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, then $\|\mathbb{E}\boldsymbol{A}\|_{\text{op}} \leq \mathbb{E}\|\boldsymbol{A}\|_{\text{op}}$. Let $\mathcal{S} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$, by using $\|\boldsymbol{B}\|_{\text{op}} = \sup_{U,V \in \mathcal{S}} U^T \boldsymbol{B} V$, we have

$$\|\mathbb{E}\boldsymbol{A}\|_{\mathrm{op}} = \sup_{U,V\in\mathcal{S}} \mathbb{E}[U^T \boldsymbol{A} V] \leq \mathbb{E}[\sup_{U,V\in\mathcal{S}} U^T \boldsymbol{A} V] = \mathbb{E}\|\boldsymbol{A}\|_{\mathrm{op}}$$

<u>Fact 3:</u> Given random matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, then $\|\mathbb{E}(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})^T(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})\|_{\text{op}} \leq \|\mathbb{E}\boldsymbol{A}^T\boldsymbol{A}\|_{\text{op}}$, $\|\mathbb{E}(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})^T\|_{\text{op}} \leq \|\mathbb{E}\boldsymbol{A}\boldsymbol{A}^T\|_{\text{op}}$.

We only show the first inequality, the second follows likewise. By calculation we have

$$\|\mathbb{E}(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})^T(\boldsymbol{A} - \mathbb{E}\boldsymbol{A})\|_{\mathrm{op}} = \|\mathbb{E}\boldsymbol{A}^T\boldsymbol{A} - \mathbb{E}\boldsymbol{A}^T\mathbb{E}\boldsymbol{A}\|_{\mathrm{op}} \le \|\mathbb{E}\boldsymbol{A}^T\boldsymbol{A}\|_{\mathrm{op}},$$

where we use the positive semi-definiteness of $\mathbb{E}A^T\mathbb{E}A$ and $\mathbb{E}A^TA - \mathbb{E}A^T\mathbb{E}A$. II. We now start the proof. We first note that

$$\boldsymbol{\Sigma}_{Y\boldsymbol{X}} = \mathbb{E}(Y_k \boldsymbol{X}_k) = \mathbb{E}(\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* \rangle \boldsymbol{X}_k + \epsilon_k \boldsymbol{X}_k) = \mathbb{E}(\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* \rangle \boldsymbol{X}_k),$$

so by using triangle inequality we obtain

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle-\gamma\cdot\dot{Y}_{\boldsymbol{k}}\right]\boldsymbol{X}_{\boldsymbol{k}}\right\|_{\mathrm{op}} \leq \left\|\frac{1}{n}\sum_{k=1}^{n}\left[\gamma\cdot\dot{Y}_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}}-\mathbb{E}(\gamma\cdot\dot{Y}_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}})\right]\right\|_{\mathrm{op}} + \left\|\mathbb{E}\left[\left(\gamma\cdot\dot{Y}_{\boldsymbol{k}}-Y_{\boldsymbol{k}}\right)\boldsymbol{X}_{\boldsymbol{k}}\right]\right\|_{\mathrm{op}} + \left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle\boldsymbol{X}_{\boldsymbol{k}}-\mathbb{E}(\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle\boldsymbol{X}_{\boldsymbol{k}})\right]\right\|_{\mathrm{op}} \coloneqq R_{1}+R_{2}+R_{3}.$$
(C.1)

<u>Bound of R_1 </u>. We intend to use matrix Bernstein inequality (See Theorem 6.1.1 in [90]) to bound R_1 . Consider a finite sequence of independent, zero-mean random matrices $\{\mathbf{S}_k := \gamma \cdot \dot{Y}_k \mathbf{X}_k - \mathbb{E}(\gamma \cdot \dot{Y}_k \mathbf{X}_k) : k \in [n]\}$, and by Fact 2 we have

$$\|\boldsymbol{S}_{\boldsymbol{k}}\|_{\mathrm{op}} \leq \|\boldsymbol{\gamma} \cdot \dot{Y}_{\boldsymbol{k}} \boldsymbol{X}_{\boldsymbol{k}}\|_{\mathrm{op}} + \|\mathbb{E}(\boldsymbol{\gamma} \cdot \dot{Y}_{\boldsymbol{k}} \boldsymbol{X}_{\boldsymbol{k}})\|_{\mathrm{op}} \leq \boldsymbol{\gamma} + \mathbb{E}\|\boldsymbol{\gamma} \cdot \dot{Y}_{\boldsymbol{k}} \boldsymbol{X}_{\boldsymbol{k}}\|_{\mathrm{op}} \leq 2\boldsymbol{\gamma}.$$

Then we bound $\max\{\|n \cdot \mathbb{E} S_k S_k^T\|_{\text{op}}, \|n \cdot \mathbb{E} S_k^T S_k\|_{\text{op}}\}$. By using Fact 1 and Fact 3. we have $\|\mathbb{E} S_k S_k^T\|_{\text{op}} \leq \|\mathbb{E}\gamma^2 \cdot X_k X_k^T\|_{\text{op}} \leq \gamma^2/d$, and similarly it holds that $\|\mathbb{E} S_k^T S_k\|_{\text{op}} \leq \gamma^2/d$. Thus, we have

$$\nu\left(\sum_{k=1}^{n} \boldsymbol{S}_{\boldsymbol{k}}\right) := \max\{\|n \cdot \mathbb{E}\boldsymbol{S}_{\boldsymbol{k}}\boldsymbol{S}_{\boldsymbol{k}}^{T}\|_{\text{op}}, \|n \cdot \mathbb{E}\boldsymbol{S}_{\boldsymbol{k}}^{T}\boldsymbol{S}_{\boldsymbol{k}}\|_{\text{op}}\} \le \frac{n\gamma^{2}}{d}.$$

By using matrix Bernstein inequality, for any t > 0 we have

$$\mathbb{P}(R_1 \ge t) \le 2d \exp\left(-\frac{nt^2}{2\gamma[\gamma/d + 2t/3]}\right).$$
(C.2)

We let $t = 2\gamma \sqrt{\frac{\delta \log(2d)}{nd}}$, when $\frac{\delta d \log(2d)}{n} < 9/16$ it holds that

$$\mathbb{P}\left(R_1 \ge 2\gamma \sqrt{\frac{\delta \log(2d)}{nd}}\right) \le 2d \exp\left(-\frac{ndt^2}{4\gamma^2}\right) \le (2d)^{1-\delta}.$$
(C.3)

<u>Bound of R2.</u> We first bound the max norm error $\|\mathbb{E}[(\gamma \cdot \dot{Y}_k - Y_k) \mathbf{X}_k]\|_{\max}$. Let $X_{k,ij}$ denotes the (i, j)-th entry of \mathbf{X}_k . Consider specific $(i, j) \in [d] \times [d]$, then the distribution of $X_{k,ij}$ is given by $\mathbb{P}(X_{k,ij} = 1) = d^{-2}$, otherwise $X_{k,ij} = 0$. Also, we let Θ_{ij}^* be the (i,j)-th entry of Θ^* . When $|Y_k| < \gamma$ by Lemma 1 we have $\mathbb{E}_{\Lambda_k}(\gamma \cdot \dot{Y}_k) = Y_k$. Furthermore, it holds that

$$\begin{aligned} |\mathbb{E}(\gamma \cdot Y_k - Y_k) X_{k,ij}| &\leq \mathbb{E}|Y_k| X_{k,ij} \mathbb{1}(|Y_k| \geq \gamma) = \mathbb{E}\left(\mathbb{E}\left[|Y_k| X_{k,ij} \mathbb{1}(|Y_k| \geq \gamma) \middle| X_{k,ij}\right]\right) \\ &= d^{-2} \mathbb{E}\left[|Y_k| X_{k,ij} \mathbb{1}(|Y_k| \geq \gamma) \middle| X_{k,ij} = 1\right] = d^{-2} \mathbb{E}\left[|\Theta_{ij}^* + \epsilon_k| \mathbb{1}(|\Theta_{ij}^* + \epsilon_k| \geq \gamma)\right]. \end{aligned}$$

By (4.5) we have $|\Theta_{ij}^*| \leq \alpha^*$, recall that $\gamma \geq 2\alpha^*$, so $|\Theta_{ij}^* + \epsilon_k| \geq \gamma$ implies

$$|\epsilon_k| \ge |\Theta_{ij}^* + \epsilon_k| - |\Theta_{ij}^*| \ge \gamma - \alpha^* \ge \frac{\gamma}{2},$$

which implies $|\epsilon_k| \ge \alpha^*$. Moreover, we obtain

$$|\Theta_{ij}^* + \epsilon_k| \le |\Theta_{ij}^*| + |\epsilon_k| \le \alpha^* + |\epsilon_k| \le 2|\epsilon_k|.$$

Thus, we apply Cauchy-Schwarz inequality, (4.12), Proposition 1, it gives

$$\begin{aligned} |\mathbb{E}(\gamma \cdot \dot{Y}_k - Y_k) X_{k,ij}| &\leq d^{-2} \mathbb{E} \left[|\Theta_{ij}^* + \epsilon_k| \mathbb{1}(|\Theta_{ij}^* + \epsilon_k| \geq \gamma) \right] &\leq 2d^{-2} \mathbb{E} \left[|\epsilon_k| \mathbb{1}(|\epsilon_k| \geq \frac{\gamma}{2}) \right] \\ &\leq 2d^{-2} \sqrt{\mathbb{E}\epsilon_k^2} \sqrt{\mathbb{P} \left(|\epsilon_k| \geq \frac{\gamma}{2} \right)} \lesssim d^{-2} \sigma \exp \left(-\frac{D_1 \gamma^2}{\sigma^2} \right) \lesssim d^{-1} \sigma \sqrt{\frac{\delta \log(2d)}{nd}}, \end{aligned}$$

where the last " \leq " follows from γ given in (4.12) with sufficiently large C_{13} . Since the estimation holds for any (i, j), we have $\|\mathbb{E}[(\gamma \dot{Y}_k - Y_k) \mathbf{X}_k]\|_{\max} \leq d^{-1}\sigma \sqrt{\frac{\delta \log(2d)}{nd}}$. By the relation between $\|.\|_{\text{op}}$ and $\|.\|_{\max}$, it further gives

$$R_2 \le d \cdot \|\mathbb{E}[(\gamma \cdot \dot{Y}_k - Y_k) \boldsymbol{X}_k]\|_{\max} \le \sigma \sqrt{\frac{\delta \log(2d)}{nd}}$$

<u>Bound of R_3 </u>. Similar to R_1 we use matrix Bernstein inequality. We consider the finite independent, zero-mean random matrix sequence

$$\left\{ \boldsymbol{W}_{\boldsymbol{k}} := \left\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta}^* \right\rangle \boldsymbol{X}_{\boldsymbol{k}} - \mathbb{E}(\left\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta}^* \right\rangle \boldsymbol{X}_{\boldsymbol{k}}) : k \in [n] \right\}.$$

Note that $|\langle X_k, \Theta^* \rangle| \leq \alpha^*$, so $||\langle X_k, \Theta^* \rangle X_k||_{op} \leq \alpha^*$, by Fact 2 we have

$$\|\boldsymbol{W}_{\boldsymbol{k}}\|_{\mathrm{op}} \leq \|\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta}^* \rangle \boldsymbol{X}_{\boldsymbol{k}}\|_{\mathrm{op}} + \|\mathbb{E} \langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta}^* \rangle \boldsymbol{X}_{\boldsymbol{k}}\|_{\mathrm{op}} \leq 2\alpha^* \leq 2\gamma.$$

By using Fact 1 and Fact 3, we obtain $\|\mathbb{E}W_k W_k^T\|_{\text{op}} \leq \|\mathbb{E}\langle X_k, \Theta^* \rangle^2 X_k X_k^T\|_{\text{op}} \leq \frac{(\alpha^*)^2}{d}$. Likewise we have $\|\mathbb{E}W_k^T W_k\|_{\text{op}} \leq \frac{(\alpha^*)^2}{d}$, so we derive the bound

$$\nu\left(\sum_{k=1}^{n} \boldsymbol{W}_{\boldsymbol{k}}\right) := \max\{\|\boldsymbol{n} \cdot \mathbb{E} \boldsymbol{W}_{\boldsymbol{k}} \boldsymbol{W}_{\boldsymbol{k}}^{T}\|_{\text{op}}, \|\boldsymbol{n} \cdot \mathbb{E} \boldsymbol{W}_{\boldsymbol{k}}^{T} \boldsymbol{W}_{\boldsymbol{k}}\|_{\text{op}}\} \le \frac{n(\alpha^{*})^{2}}{d} \le \frac{n\gamma^{2}}{d}.$$

Parallel to R_1 , by using Matrix Bernstein inequality and set $t = 2\gamma \sqrt{\frac{\delta \log(2d)}{nd}}$,

$$\mathbb{P}\left(R_3 \ge 2\gamma \sqrt{\frac{\delta \log(2d)}{nd}}\right) \le (2d)^{1-\delta}.$$
(C.4)

We combine the obtained upper bounds for R_1, R_2, R_3 and draw the conclusion that with probability higher than $1 - 2d^{1-\delta}$, we have

$$\left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X_{k}},\boldsymbol{\Theta^{*}}\right\rangle-\gamma\cdot\dot{Y}_{k}\right]\boldsymbol{X_{k}}\right\|_{\mathrm{op}}\lesssim\gamma\sqrt{\frac{\delta\log d}{nd}}.$$

Now we can use $\gamma \leq C_{13} \max\{\alpha^*, \sigma\} \sqrt{\log n}$ to conclude the proof.

Proof of Lemma 4. I. We first decompose the complementary event of (4.15) which can be stated as $\mathscr{B} = \{\exists \Theta_0 \in \mathcal{C}(\psi), \text{s.t. } \mathcal{F}_{\mathscr{X}}(\Theta_0) \leq \kappa d^{-2} \|\Theta_0\|_F^2 - T_0\}$. Note that $\mathbb{E}\mathcal{F}_{\mathscr{X}}(\Theta) = d^{-2} \|\Theta\|_F^2$, so \mathscr{B} implies the following event

$$\{\exists \Theta_{\mathbf{0}} \in \mathcal{C}(\psi), \text{s.t.} | \mathcal{F}_{\mathscr{X}}(\Theta_{\mathbf{0}}) - \mathbb{E}\mathcal{F}_{\mathscr{X}}(\Theta_{\mathbf{0}})| \ge (1-\kappa)d^{-2} \|\Theta_{\mathbf{0}}\|_{\mathrm{F}}^{2} + T_{0}\}.$$
 (C.5)

Let $D_0 = (\alpha^* d)^2 (\psi \delta \log(2d)/n)^{1/2}$, then by (4.14) we have $\|\Theta_0\|_F^2 \ge D_0$, so by a specific $\beta > 1$ (that will be selected later), there exists positive integer l such that $\|\Theta_0\|_F^2 \in [\beta^{l-1}D_0, \beta^l D_0)$. We further consider $\mathcal{C}(\psi, l) = \mathcal{C}(\psi) \cap \{\Theta : \|\Theta\|_F^2 \in [\beta^{l-1}D_0, \beta^l D_0)\}$, and define a term

$$\mathcal{Z}_{\mathscr{X}}(l) = \sup_{\mathbf{\Theta}\in\mathcal{C}(\psi,l)} |\mathcal{F}_{\mathscr{X}}(\mathbf{\Theta}) - \mathbb{E}\mathcal{F}_{\mathscr{X}}(\mathbf{\Theta})|,$$

then we know the event defined in (C.5) implies the event

$$\mathscr{B}_l = \{ \mathscr{Z}_{\mathscr{X}}(l) \ge (1-\kappa)d^{-2}\beta^{l-1}D_0 + T_0 \}.$$
(C.6)

By taking the union bound over $l \in \mathbb{N}^*$ we obtain $\mathbb{P}(\mathscr{B}) \leq \sum_{l=1}^{\infty} \mathbb{P}(\mathscr{B}_l)$.

II. It suffices to bound $\mathbb{P}(\mathscr{B}_l)$. We first bound the deviation $|\mathcal{Z}_{\mathscr{X}}(l) - \mathbb{E}\mathcal{Z}_{\mathscr{X}}(l)|$. We consider $\widetilde{\mathscr{X}} = (\widetilde{X}_1, X_2, ..., X_n)$ where only the first component may be different from \mathscr{X}

$$\sup_{\mathscr{X},\widetilde{\mathscr{X}}} |\mathcal{Z}_{\mathscr{X}}(l) - \mathcal{Z}_{\widetilde{\mathscr{X}}}(l)| = \sup_{\mathscr{X},\widetilde{\mathscr{X}}} \left| \sup_{\Theta \in \mathcal{C}(\psi,l)} |\mathcal{F}_{\mathscr{X}}(\Theta) - \mathbb{E}\mathcal{F}_{\mathscr{X}}(\Theta)| - \sup_{\Theta \in \mathcal{C}(\psi,l)} |\mathcal{F}_{\widetilde{\mathscr{X}}}(\Theta) - \mathbb{E}\mathcal{F}_{\widetilde{\mathscr{X}}}(\Theta)| \right|$$

$$\leq \sup_{\mathscr{X},\widetilde{\mathscr{X}}} \left| \sup_{\Theta \in \mathcal{C}(\psi,l)} |\mathcal{F}_{\mathscr{X}}(\Theta) - \mathcal{F}_{\widetilde{\mathscr{X}}}(\Theta)| \right| = \sup_{X_{1},\widetilde{X}_{1}} \sup_{\Theta \in \mathcal{C}(\psi,l)} \frac{1}{n} |\langle X_{1}, \Theta \rangle|^{2} - |\langle \widetilde{X}_{1}, \Theta \rangle|^{2} | \leq \frac{4(\alpha^{*})^{2}}{n}.$$

Note that n components of \mathscr{X} are symmetrical, by bounded different inequality (e.g., Corollary 2.21, [94]), for any t > 0 we have

$$\mathbb{P}\Big(\mathcal{Z}_{\mathscr{X}}(l) - \mathbb{E}\mathcal{Z}_{\mathscr{X}}(l) \ge t\Big) \le \exp\Big(-\frac{nt^2}{8(\alpha^*)^4}\Big).$$
(C.7)

It remains to bound $\mathbb{E}\mathcal{Z}_{\mathscr{X}}(l)$. Let $\mathscr{E} = (\varepsilon_1, ..., \varepsilon_n)$ be i.i.d. Rademacher random variables satisfying $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$, then by symmetrization of expectations (e.g., Theorem 16.1, [91]), Talagrand's inequality (e.g., Theorem 16.2, [91]), the second constraint

in (4.14), it yields that

$$\begin{split} & \mathbb{E}\mathcal{Z}_{\mathscr{X}}(l) = \mathbb{E}\sup_{\Theta \in \mathcal{C}(\psi, l)} |\mathcal{F}_{\mathscr{X}}(\Theta) - \mathbb{E}\mathcal{F}_{\mathscr{X}}(\Theta)| = \mathbb{E}\sup_{\Theta \in \mathcal{C}(\psi, l)} \left|\frac{1}{n} \sum_{k=1}^{n} \left\{\left\langle \boldsymbol{X}_{\boldsymbol{k}}, \Theta\right\rangle^{2} - \mathbb{E}\left\langle \boldsymbol{X}_{\boldsymbol{k}}, \Theta\right\rangle^{2}\right\}\right| \\ & \leq 2\mathbb{E}_{\mathscr{X}} \mathbb{E}_{\mathscr{E}}\sup_{\Theta \in \mathcal{C}(\psi, l)} \left|\frac{1}{n} \sum_{k=1}^{n} \varepsilon_{k} \left\langle \boldsymbol{X}_{\boldsymbol{k}}, \Theta\right\rangle^{2}\right| \leq 16\alpha^{*} \mathbb{E}\sup_{\Theta \in \mathcal{C}(\psi, l)} \left|\left\langle\frac{1}{n} \sum_{k=1}^{n} \varepsilon_{k} \boldsymbol{X}_{\boldsymbol{k}}, \Theta\right\rangle\right| \\ & \leq 16\alpha^{*} \mathbb{E}\left\|\frac{1}{n} \sum_{k=1}^{n} \varepsilon_{k} \boldsymbol{X}_{\boldsymbol{k}}\right\|_{\text{op}} \sup_{\Theta \in \mathcal{C}(\psi, l)} \|\Theta\|_{\text{nu}} \leq 160\alpha^{*} r^{\frac{1}{2-q}} \{\beta^{l} D_{0}\}^{\frac{1-q}{2-q}} \mathbb{E}\left\|\frac{1}{n} \sum_{k=1}^{n} \varepsilon_{k} \boldsymbol{X}_{\boldsymbol{k}}\right\|_{\text{op}}. \end{split}$$

$$(C.8)$$

Assume $d \log(2d)/n < 1/16$, by matrix bernstein inequality (Theorem 6.1.1, [90]) it holds that $\mathbb{E} \left\| \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \boldsymbol{X}_{\boldsymbol{k}} \right\|_{\text{op}} \leq \frac{3}{2} \sqrt{\frac{\log(2d)}{nd}}$. We then plug it in (C.8), some algebra yields

$$\mathbb{E}\mathcal{Z}_{\mathscr{X}}(l) \le \left\{ (2-q)T_0 \right\}^{\frac{1}{2-q}} \left\{ d^{-2}\beta^l D_0 \right\}^{\frac{1-q}{2-q}} \le \frac{1-q}{2-q} \frac{\beta^l D_0}{d^2} + T_0 \tag{C.9}$$

By combining with (C.6), (C.7) and let $\kappa_1 = \frac{1-\kappa}{\beta} - \frac{1-q}{2-q}$ (here we assume $\kappa_1 \in (0, 1)$ since we can choose κ sufficiently close to 0, β sufficiently close to 1), we have

$$\mathbb{P}(\mathscr{B}_l) \le \mathbb{P}\Big(\mathcal{Z}_{\mathscr{X}}(l) - \mathbb{E}\mathcal{Z}_{\mathscr{X}}(l) \ge \kappa_1 \frac{\beta^l D_0}{d^2}\Big) \le \exp\Big(-\frac{n\kappa_1^2 \beta^{2l} D_0^2}{8(\alpha^* d)^4}\Big).$$
(C.10)

We further plug in D_0 and use $\beta^{2l} \ge 2l \log \beta$, it yields that

$$\mathbb{P}(\mathscr{B}) \le \sum_{l=1}^{\infty} \mathbb{P}(\mathscr{B}_l) \le \sum_{l=1}^{\infty} \left[(2d)^{-\frac{\psi \delta \kappa_1^2 \log \beta}{4}} \right]^l \le d^{-\delta},$$

the last inequality holds since we can let ψ be large such that $\psi \ge 4(\kappa_1^2 \log \beta)^{-1}$.

Proof of Theorem 12. I. By Lemma 3 we can choose sufficiently large C_{14} in (4.17) to ensure (4.7) holds with probability higher than $1-2d^{1-\delta}$, then (4.8) holds with high probability. From Lemma 4 we can further rule out probability $d^{-\delta}$ to ensure (4.15) holds.

By (4.5) and (4.6) we have $\|\widehat{\boldsymbol{\Delta}}\|_{\max} \leq \|\widehat{\boldsymbol{\Theta}}\|_{\max} + \|\boldsymbol{\Theta}^*\|_{\max} \leq 2\alpha^*$. Thus, the estimation error satisfies the first constraint of $\mathcal{C}(\psi)$. Since (4.8) displays the second constraint in $\mathcal{C}(\psi)$, whether $\widehat{\boldsymbol{\Delta}} \in \mathcal{C}(\psi) \in \mathcal{C}(\psi)$ holds only depends on the third constraint, and let us discuss as follows:

<u>Case 1</u>. $\widehat{\Delta} \notin \mathcal{C}(\psi)$. Note that it can only violate the third constraint of $\mathcal{C}(\psi)$, so we know that $\|\widehat{\Delta}\|_{\mathrm{F}}^2 \leq (\alpha^* d)^2 \sqrt{\frac{\psi \delta \log(2d)}{n}}$. Under the assumption $r \gtrsim d^q$, $n \lesssim d^2 \log(2d)$, it holds that

$$\frac{\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^2}{d^2} \lesssim (\alpha^*)^2 \sqrt{\frac{\delta \log(2d)}{n}} \lesssim (\alpha^*)^2 \frac{\delta d \log d}{n} \lesssim r d^{-q} \left((\alpha^*)^2 \frac{\delta d \log d}{n} \right)^{1-q/2}.$$
 (C.11)

<u>Case 2</u>. $\widehat{\Delta} \in \mathcal{C}(\psi)$. By (4.15) we know $\mathcal{F}_{\mathscr{X}}(\widehat{\Delta}) \geq \kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2 - T_0$. If $T_0 \geq \frac{1}{2}\kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2$, then we plug in T_0 and obtain

$$\frac{\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^2}{d^2} \lesssim r d^{-q} \left((\alpha^*)^2 \frac{\delta d \log d}{n} \right)^{1-q/2}.$$
 (C.12)

If $T_0 \leq \frac{1}{2}\kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2$, then we have $\mathcal{F}_{\mathscr{X}}(\widehat{\Delta}) \geq \frac{1}{2}\kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2$. Note that this displays the RSC in (4.9), so we now use Corollary 3 and obtain

$$\frac{\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^2}{d^2} \lesssim r d^{-q} \Big(\max\{(\alpha^*)^2, \sigma^2\} \log d \log n \frac{\delta d}{n} \Big)^{1-q/2}.$$
(C.13)

Now we can see that in all cases considered above, the bound of $\|\widehat{\Delta}\|_{\rm F}^2/d^2$ in (4.18) holds. Then a direct application of (4.8) delivers the bound of $\|\widehat{\Delta}\|_{\rm nu}/d$ in (4.18). To conclude, (4.18) holds with probability higher than $1 - 3d^{1-\delta}$.

C.2 Heavy-tailed Data

Proof of Lemma 5. From (4.1) we have $\mathbb{E}(Y_k X_k) = \mathbb{E}(\langle X_k, \Theta^* \rangle X_k)$, and since $\eta < \gamma$ by Lemma 1 we know $\mathbb{E}_{\Lambda_k}(\gamma \cdot \dot{Y}_k) = \tilde{Y}_k$, hence we have

$$\begin{split} & \left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle-\gamma\cdot\dot{Y}_{\boldsymbol{k}}\right]\boldsymbol{X}_{\boldsymbol{k}}\right\|_{\mathrm{op}} \leq \left\|\frac{1}{n}\sum_{k=1}^{n}\left[\gamma\cdot\dot{Y}_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}}-\mathbb{E}(\gamma\cdot\dot{Y}_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}})\right]\right\|_{\mathrm{op}} + \\ & \left\|\mathbb{E}\left[\left(\widetilde{Y}_{\boldsymbol{k}}-Y_{\boldsymbol{k}}\right)\boldsymbol{X}_{\boldsymbol{k}}\right]\right\|_{\mathrm{op}} + \left\|\frac{1}{n}\sum_{k=1}^{n}\left[\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle\boldsymbol{X}_{\boldsymbol{k}}-\mathbb{E}(\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{\Theta}^{*}\right\rangle\boldsymbol{X}_{\boldsymbol{k}})\right]\right\|_{\mathrm{op}} := R_{1}+R_{2}+R_{3}. \end{split}$$

<u>Bound of R_1, R_3 </u>. We use matrix Bernstein inequality (Theorem 6.1.1, [90]), and the arguments are exactly the same as the corresponding parts in the proof of Lemma 3. As a result, one can still invoke Matrix Bernstein to show (C.3) and (C.4), but only with different value of γ . To obtain the explicit form of the bounds, we further plug in γ in (4.20), with probability higher than $1 - 2d^{1-\delta}$ it gives

$$\max\{R_1, R_3\} \lesssim \max\{\alpha^*, \sqrt{M}\} \left(\frac{\delta \log d}{nd^3}\right)^{1/4}$$

<u>Bound of R₂.</u> Let $X_{k,ij}$ be the (i, j)-th entry of \mathbf{X}_k , where $(i, j) \in [d] \times [d]$ is fixed, we first bound the element-wise error $|\mathbb{E}(\tilde{Y}_k - Y_k)X_{k,ij}|$. Recall the definition of truncation, Lemma 1 gives

$$\left|\mathbb{E}\left[(\widetilde{Y}_{k}-Y_{k})X_{k,ij}\right]\right|=\left|\mathbb{E}\left[(\widetilde{Y}_{k}-Y_{k})X_{k,ij}\mathbb{1}(|Y_{k}|>\eta)\right]\right|\leq\mathbb{E}|Y_{k}|X_{k,ij}\mathbb{1}(|Y_{k}|>\eta).$$

Note that $X_{k,ij}$ can only be 1 or 0, and $\mathbb{P}(X_{k,ij} = 1) = d^{-2}$. Let Θ_{ij}^* be the (i, j)-th entry of Θ^* , we further compute it via law of total expectation, then use Cauchy-Schwarz inequality and Marcov's inequality, finally plug in η finally. These steps deliver

$$\mathbb{E}|Y_{k}|X_{k,ij}\mathbb{1}(|Y_{k}| > \eta) = \mathbb{E}\left(\mathbb{E}\left[|Y_{k}|X_{k,ij}\mathbb{1}(|Y_{k}| > \eta) \middle| X_{k,ij}\right]\right)$$

$$= d^{-2}\mathbb{E}\left[|Y_{k}|X_{k,ij}\mathbb{1}(|Y_{k}| > \eta) \middle| X_{k,ij} = 1\right] = d^{-2}\mathbb{E}\left[|\Theta_{ij}^{*} + \epsilon_{k}|\mathbb{1}(|\Theta_{ij}^{*} + \epsilon_{k}| \ge \eta)\right]$$

$$\leq d^{-2}\sqrt{\mathbb{E}|\Theta_{ij}^{*} + \epsilon_{k}|^{2}\mathbb{P}(|\Theta_{ij}^{*} + \epsilon_{k}| \ge \eta)} \leq d^{-2}\eta^{-1}\mathbb{E}|\Theta_{ij}^{*} + \epsilon_{k}|^{2}$$

$$\leq 2d^{-2}\eta^{-1}[(\Theta_{ij}^{*})^{2} + \mathbb{E}\epsilon_{k}^{2}] \leq 4(dC_{15})^{-1}\max\{\alpha^{*}, \sqrt{M}\}\left(\frac{\delta\log d}{nd^{3}}\right)^{1/4}$$

Since the above analysis works for all $(i, j) \in [d] \times [d]$, this is also an upper bound for the max norm, which delivers a bound for operator norm

$$R_2 \le d \left\| \mathbb{E}\left[(\widetilde{Y}_k - Y_k) \boldsymbol{X}_k \right] \right\|_{\max} \lesssim \max\{\alpha^*, \sqrt{M}\} \left(\frac{\delta \log d}{n d^3} \right)^{1/4}.$$
(C.14)

The result follows from the upper bounds for R_1, R_2, R_3 .

Proof of Theorem 13. By Lemma 5 we can choose sufficiently large C_{17} in (4.22) to ensure (4.7) holds with probability higher than $1 - 2d^{1-\delta}$, then it further implies (4.8), meaning that $\widehat{\Delta}$ satisfies the second constraint of $\mathcal{C}(\psi)$. From Lemma 4 we can further rule out probability $d^{-\delta}$ so that (4.15) holds. Evidently we have $\|\widehat{\Delta}\|_{\max} \leq 2\alpha^*$. Thus, with probability higher than $1 - 3d^{1-\delta}$, $\widehat{\Delta}$ satisfies the first two constraints of $\mathcal{C}(\psi)$, and (4.15) holds. Based on these conditions we further discuss as follows:

<u>Case 1.</u> $\Delta \notin C(\psi)$, then by exactly the same analysis in proof of Theorem 12, we obtain (C.11).

<u>Case 2.</u> $\widehat{\Delta} \in \mathcal{C}(\psi)$, then we have $\mathcal{F}_{\mathscr{X}}(\widehat{\Delta}) \geq \kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2 - T_0$. If $T_0 \geq \frac{1}{2}\kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2$, then (C.12) holds. Otherwise, we have the restricted strong convexity $\mathcal{F}_{\mathscr{X}}(\widehat{\Delta}) \geq \frac{1}{2}\kappa d^{-2} \|\widehat{\Delta}\|_{\mathrm{F}}^2$. We then apply Corollary 3 and plug in λ , it holds that

$$\|\widehat{\boldsymbol{\Delta}}\|_{\mathrm{F}}^2/d^2 \lesssim r d^{-q} \Big(\max\{(\alpha^*)^2, M\} \sqrt{\frac{\delta d \log d}{n}}\Big)^{1-q/2}.$$
 (C.15)

It is not hard to see that the right hand side of (C.15) dominates the bound in (C.11) and (C.12), so the bound for $\|\widehat{\Delta}\|_{\mathrm{F}}^2/d^2$ in (4.23) holds. The bound for $\|\widehat{\Delta}\|_{\mathrm{nu}}/d$ follows from a direct application of (4.8).

Proof of Theorem 14. We simply write $\mathscr{K}(r, \alpha^*)$ as \mathscr{K} . Recall our choice of dithering scale

$$\gamma \asymp (\alpha^* + \sqrt{M}) \left(\frac{n}{d\log d}\right)^{1/4}.$$

We define $\hat{\alpha} = \min\left\{\frac{1}{4}\alpha^*, D_1\gamma\sqrt{\frac{rd}{n}}\right\}$, then by invoking [34, Lemma A.3] (set γ therein to 1), we can find $\mathscr{K}_1 = \{\Theta^{(1)}, \Theta^{(2)}, ..., \Theta^{(N)}\} \subset \mathscr{K}$ such that:

- $N \ge \exp\left(\frac{rd}{16}\right);$
- $\mathscr{K}_1 \subset \{-\hat{\alpha}, \hat{\alpha}\}^{d \times d}$, i.e., entries of $\Theta \in \mathscr{K}_1$ are either $\hat{\alpha}$ or $-\hat{\alpha}$;
- for different $\Theta^{(i)}, \Theta^{(j)} \in \mathscr{K}_1$, it holds that $\|\Theta^{(i)} \Theta^{(j)}\|_F^2 > \frac{1}{2}(\hat{\alpha}d)^2$.

It is sufficient to deal with the noiseless case $\epsilon_k = 0$, i.e., when the underlying matrix is Θ , we let $\mathsf{T}_{\eta}(a) = \operatorname{sign}(a) \min\{|a|, \eta\}$ be the truncation operator and have the observations

$$\mathscr{P}(\boldsymbol{\Theta}) = \{ \dot{Y}_k = \operatorname{sign} \left(\mathsf{T}_{\eta}(\left\langle \boldsymbol{X}_k, \boldsymbol{\Theta} \right\rangle) + \Lambda_k \right), k = 1, 2, ..., n \}$$

By reduction to hypothesis testing and Fano's inequality [82, Section 4], we have

$$\inf_{\widehat{\Theta}} \sup_{\Theta} \mathbb{P}_{\Theta} \left(\| \widehat{\Theta} - \Theta \|_{F}^{2} > \frac{1}{8} (\hat{\alpha}d)^{2} \right) \geq 1 - \frac{\frac{1}{N^{2}} \sum_{i,j=1}^{N} \mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)})) + \log 2}{\log N}.$$
(C.16)

We now estimate $\mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)}))$. Let $\Theta^{(i)}$ be the underlying parameter, because $\langle \mathbf{X}_k, \Theta^{(i)} \rangle = \pm \hat{\alpha}$, and also $\eta > \hat{\alpha}$ holds trivially, so $\dot{Y}_k = \operatorname{sign}(\langle \mathbf{X}_k, \Theta^{(i)} \rangle + \Lambda_k)$ follows a (symmetrized) Bernoulli distribution with success probability

$$\mathbb{P}(\dot{Y}_k = 1) = \mathbb{P}(\Lambda_k > -\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^{(i)} \rangle) = \frac{\gamma \mp \hat{\alpha}}{2\gamma} = \frac{1}{2} \mp \frac{\hat{\alpha}}{2\gamma}.$$

Moreover, because $\mathscr{P}(\Theta^{(i)})$ consists of n i.i.d. observations,

$$\mathsf{KL}(\mathscr{P}(\Theta^{(i)}), \mathscr{P}(\Theta^{(j)})) \le n \cdot \mathsf{KL}\Big(\frac{1}{2} + \frac{\hat{\alpha}}{2\gamma}, \frac{1}{2} - \frac{\hat{\alpha}}{2\gamma}\Big),$$

where we use $\mathsf{KL}(p,q)$ to denote the KL divergence between Bernoulli distribution with success probability p and q. Because $|\frac{\hat{\alpha}}{2\gamma}| \leq \frac{1}{4} \frac{\alpha^*}{2\gamma} \leq \frac{1}{8}$ (recall that we assume $\gamma > \eta > \alpha^*$), by using [34, Lemma A.4] we obtain

$$\mathsf{KL}(\mathscr{P}(\boldsymbol{\Theta}^{(i)}), \mathscr{P}(\boldsymbol{\Theta}^{(j)})) \le \frac{n(\hat{\alpha}/\gamma)^2}{(\frac{1}{2} + \frac{\hat{\alpha}}{2\gamma})(\frac{1}{2} - \frac{\hat{\alpha}}{2\gamma})} \le \frac{16n\hat{\alpha}^2}{3\gamma^2} \le \frac{16D_1^2rd}{3} < \frac{rd}{150}, \tag{C.17}$$

note that the last two inequalities holds because $\hat{\alpha} \leq D_1 \gamma \sqrt{\frac{rd}{n}}$ and we can select sufficiently small D_1 . Now we put this into (C.16). Because $\log N \geq \frac{rd}{16}$, with slightly large rd to overcome log 2, the probability term on the right hand side of (C.16) is greater than $\frac{3}{4}$, while the event on the left hand side is just

$$\|\widehat{\Theta} - \Theta\|_{F}^{2}/d^{2} > \frac{1}{8}\hat{\alpha}^{2} = \min\left\{\frac{(\alpha^{*})^{2}}{128}, D_{2}((\alpha^{*})^{2} + M)r\sqrt{\frac{d}{n\log d}}\right\},\$$

where we plug in the value of γ . The proof is complete.

D Comparisons with Related Work

D.1 1-bit Compressed Sensing

In this part we compare our Theorems 9-10 with existing results of 1-bit CS.

The traditional setting of 1-bit CS, where one aims to recovery a sparse d-dimensional signal Θ^* based on measurement $\dot{Y}_k = \text{sign}(X_k^T \Theta^*)$ with some X_k , was first introduced in [15] and widely studied in subsequent works (e.g., [55, 74, 75]). By projection-based method [77] or K-Lasso [76], similar results were obtained for a model with more general observation (that involves possibly unknown nonlinearity) and signal structure. Nevertheless, all these results are restricted to Gaussian sensing vectors that can be unrealistic in practice⁵. There does exist one work, [1], presents result for X_k with i.i.d. sub-Gaussian entries. However, the result in [1] is still overly restrictive and impractical, see the discussions in [40].

To overcome the restriction of Gaussian sensing vector (and also some other limitations), it was recently realized that introducing dithering noise can help. With dithering noise Λ_k ,

⁵More precisely, [76] handles $X_k \sim \mathcal{N}(0, \Sigma)$ with unknown Σ while other several papers above assume $X_k \sim \mathcal{N}(0, I_d)$.

the measurement now becomes $\dot{Y}_k = \operatorname{sign}(X_k^T \Theta^* + \Lambda_k)$. In this setting, specifically, we can recover the signal with norm information [63], achieve exponentially-decaying error rate (This requires adaptive dithering) [5], and perhaps more prominently, accommodate non-Gaussian X_k [39, 40, 89]. In what follows, we will focus on comparing Theorems 9-10 with the most relevant works [40,89] that adopt uniform dithering noise. The comparisons will be conducted on exactly sparse Θ^* since [40,89] do not adopt the formulation $\sum_{k=1}^{n} |\theta_k^*|^q \leq s, q \in (0,1)$ for approximately sparse Θ^* . For other developments on 1-bit CS (or more generally, quantized compressed sensing), we refer readers to the survey papers [16,37].

Dirksen and Mendelson [40] first essentially extend 1-bit CS to non-Gaussian X_k . Their methodology is based on random hyperplane tessellation. Specifically, under sub-Gaussian or even heavy-tailed X_k^6 with uniform dithering noise, they show a relatively small number of random hyperplanes (that depends on the complexity of Θ^*) leads to ρ -uniform tessellation on the signal set of interest. Moreover, they apply the new hyperplane tessellation results to 1-bit CS and propose two reconstruction optimization problems

$$\begin{cases} (a): \widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \sum_{k=1}^n \mathbb{1}(\operatorname{sign}(X_k^T \Theta + \Lambda_k) \neq Y_k), \quad \text{s.t.} \quad \|\Theta\|_0 \le s, \ \|\Theta\|_2 \le 1 \\ (b): \widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \quad \frac{1}{2\lambda} \|\Theta\|_2^2 - \frac{1}{2n} \sum_{k=1}^n Y_k X_k^T \Theta, \quad \text{s.t.} \quad \|\Theta\|_1 \le s, \ \|\Theta\|_2 \le 1 \end{cases}$$
(D.1)

Although (a) is shown to possess uniform recovery guarantee with fast rate in both sub-Gaussian and heavy-tailed X_k , it is essentially intractable due to the ℓ_0 constraint and the 0-1 objective function. Also, a secondary drawback is that, the information of Λ_k is needed in problem (a), which induces undesired memory or transmission costs. For these reasons, (a) is mainly of theoretical interest. Then, (b) is proposed as a convex relaxation to remedy the downside of (a). Under sub-Gaussian X_k and ϵ_k , the error rate of (b) for *s*-sparse Θ^* was shown to be $\tilde{O}(\sqrt[4]{\frac{S}{n}})$, and this is inferior to the near optimal rate $\tilde{O}(\sqrt{\frac{S}{n}})$ provided by our Theorem 9. Indeed, their error rate for sub-Gaussian data is even worse than our rate for heavy-tailed data, i.e., $\tilde{O}(\sqrt[3]{\frac{s^2}{n}})$ given in Theorem 10, while the guarantee of (b) under heavy-tailed X_k has not yet been established. On the other hand, the advantages of [40] is that their guarantee is uniform, as an outcome of the hyperplane tessellation method. They also obtain a partial extension to structured random measurement matrix in the companion work [39]. These two aspects are left as future research directions of our theories.

A result directly comparable to our Theorem 9 is due to Thrampoulidis and Rawat [89]. Under almost the same setting they assume $\Theta^* \in \mathcal{T}$ and consider a constrained Lasso

$$\widehat{\Theta} \in \underset{\Theta \in \mathcal{T}}{\operatorname{arg\,min}} \ \frac{1}{2n} \sum_{k=1}^{n} \left(X_{k}^{T} \Theta - \gamma \cdot \dot{Y}_{k} \right)^{2}.$$
(D.2)

This is analogous to our convex programming problem (3.24): up to constant, the objective of (D.2) equals the loss function (i.e., the first two terms) in (3.24); the only difference is that, the structure of Θ^* , specifically sparsity, is incorporated into (D.2) via the constraint, but appears in (3.24) as a regularizer. The relevant result is in [89, Theorem IV.1]. Interestingly, when specialized to exactly s-sparse Θ^* , they choose $\mathcal{T} = \{\Theta : \|\Theta\|_1 \leq \|\Theta^*\|_1\}$ and

⁶In [40], heavy-tailed X_k is assumed to satisfy $\mathbb{E}(|v^T X_k|^2) \leq L(\mathbb{E}|v^T X_k|)^2$ for any $v \in \mathbb{R}^d$.

show ℓ_2 norm error rate $\tilde{O}(\sqrt{\frac{s}{n}})$ that coincides with Theorem 9. Despite these similarities, our Theorem 9 exhibits several obvious improvements. Firstly, we consider pre-quantization noise ϵ_k while they only study noiseless case. Secondly, we assume zero-mean X_k satisfies $\lambda_{\min}(\Sigma_{XX}) = \Omega(1)$, but [89] requires symmetric X_k to satisfy a nondegeneracy condition formulated as $\inf_{\|v\|_2=1} \mathbb{E}|v^T X_k| = \Omega(1)$, which is more restrictive. Thirdly, their guarantee is valid with probability at least 0.99, while our probability term $1 - O(d^{2-\delta})$ is finer. In addition, we comment that a pre-estimation of $\|\Theta^*\|_1$ is needed to specify \mathcal{T} in (D.2), while our unconstrained program (3.24) is free of this issue and hence more practically appealing. On the other hand, their advantage is the more general assumption on signal structure (i.e., $\Theta^* \in T$).

We also compare with two less related works [5, 63], where the authors study $Y_k = \operatorname{sign}(X_k^T \Theta^* + \Lambda_k)$ with Gaussian dithering noise Λ_k . Convex programming problems are proposed in [5, 63] to recover *s*-sparse Θ^* , but their results are only valid for standard Gaussian X_k . Specifically, the theoretical rate in [63, Theorem 4] reads $\tilde{O}(\sqrt[5]{\frac{s}{n}})$, while [5, Theorem 2] provides an error rate $\tilde{O}(\sqrt[4]{\frac{s}{n}})$. Note that both are slower than the rates presented in our Theorem 9 (sub-Gaussian X_k), Theorem 10 (heavy-tailed X_k).

Therefore, our 1-bit CS results improve on the prior ones in terms of generality of sensing vectors and random noise (that can be sub-Gaussian or heavy-tailed), convergence rate. In particular, while [40, Theorem 1.11] involves an intractable program and a parameter hard to control (see $E(T_r)$ therein), our Theorem 10 provides the first convex program with rigorous recovery guarantee for 1-bit CS under heavy-tailed sensing vector.

D.2 1-bit Matrix Completion

In this part, we compare our Theorems 12-13 with existing results on 1-bit matrix completion (1-bit MC), a problem first proposed and studied in [23, 34]. Unlike 1-bit CS where it is still possible to recover the direction $\Theta^*/||\Theta^*||_2$ from the directly quantized measurement $\dot{Y}_k = \text{sign}(X_k^T\Theta^*)$, due to the nature of the covariate in matrix completion (i.e., $X_k = e_{i(k)}e_{j(k)}^T$), 1-bit MC could be extremely ill-posed if we only observe $\dot{Y}_k = \text{sign}(\langle X_k, \Theta^* \rangle)$. This issue happens even when Θ^* is rank-1, see the discussion in [34]. Thus, dithering noise (denoted by Λ_k) is indispensable for the well-posedness of 1-bit MC, hence the observation becomes

$$\dot{Y}_k = \operatorname{sign}(\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* \rangle + \Lambda_k).$$

Existing works consider dithering noise with rather general distribution, but particularly focus on the Logistic model and Probit model⁷.

Let us give a brief review of existing results. In the first study of 1-bit MC [34], Davenport et al. proposed to recover Θ^* via negative log-likelihood minimization (put $d_1 = d_2 = d$)

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^{d \times d}}{\operatorname{arg\,min}} \ \mathcal{L}_{\mathrm{NLL}}(\Theta), \quad \text{s.t.} \ \|\Theta\|_{\max} \le \alpha^*, \ \|\Theta\|_{\mathrm{nu}} \le \alpha^* d\sqrt{r} \ . \tag{D.3}$$

In (D.3), the first constraint is commonly used in matrix completion (see the interpretation at the beginning of Section 4), the second constraint relaxes $rank(\Theta) \leq r$ via the relation

 $\|\mathbf{\Theta}\|_{\mathrm{nu}} \leq \sqrt{\mathrm{rank}(\mathbf{\Theta})} \|\mathbf{\Theta}\|_{\mathrm{F}} \leq d \|\mathbf{\Theta}\|_{\mathrm{max}} \sqrt{\mathrm{rank}(\mathbf{\Theta})},$

⁷The Probit model corresponds to Gaussian dithering noise $\Lambda_k \sim \mathcal{N}(0, \sigma^2)$.

and the loss function $\mathcal{L}_{\text{NLL}}(\Theta)$ is

$$\mathcal{L}_{\text{NLL}}(\boldsymbol{\Theta}) = -\frac{1}{n} \sum_{k=1}^{n} \left[\mathbb{1}(\dot{Y}_{k} = 1) \log \mathbb{P}(\langle \boldsymbol{X}_{k}, \boldsymbol{\Theta} \rangle + \Lambda_{k} \ge 0) + \mathbb{1}(\dot{Y}_{k} = -1) \log \mathbb{P}(\langle \boldsymbol{X}_{k}, \boldsymbol{\Theta} \rangle + \Lambda_{k} < 0) \right].$$
(D.4)

Some developments can be seen in subsequent works, to name a few, [23] used another surrogate of matrix rank rather than the nuclear norm⁸, [10, 61, 67] extended 1-bit MC to finite alphabets, [9,10,73] imposed (exactly) low-rank constraint without relaxation, [61,67] adopted nuclear norm penalty to avoid the pre-estimation of $\|\Theta^*\|_{nu}$ needed in (D.3).

We emphasize that all above works are restricted to a noiseless setting; by saying this, we do not regard Λ_k as a detrimental noise since the dithering is indeed beneficial to the recovery. We are aware of only two recent papers [48,86] that deal with the noisy setting. In [48], Gao et al. considered a deterministic sparse pattern S^* mixing with the desired low-rank structure Θ^* . Specifically, this more general "low-rank plus sparse" model can be formulated as

$$\dot{Y}_k = \operatorname{sign}(\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* + \boldsymbol{S}^* \rangle + \Lambda_k), \text{ where } \|\operatorname{vec}(\boldsymbol{S}^*)\|_0 \le s.$$
 (D.5)

In [86], Shen et al. studied 1-bit MC with post-quantization noise in a form of sign flipping, which can be described by

$$\dot{Y}_k = \delta_k \cdot \operatorname{sign}(\langle \boldsymbol{X}_k, \boldsymbol{\Theta}^* \rangle + \Lambda_k), \text{ where } \mathbb{P}(\delta_k = -1) = \tau_0, \mathbb{P}(\delta_k = 1) = 1 - \tau_0.$$
 (D.6)

Evidently, for (D.5) or (D.6), as done in [48,86], the recovery can still be based on negative log-likelihood minimization. However, if we consider pre-quantization noise ϵ_k with unknown distribution (this is a natural and well-studied situation in other statistical estimation problems), i.e.,

$$\dot{Y}_{k} = \operatorname{sign}(\langle \boldsymbol{X}_{k}, \boldsymbol{\Theta}^{*} \rangle + \epsilon_{k} + \Lambda_{k}),$$
 (D.7)

recovery based on likelihood no longer works due to lack of knowledge on $\mathcal{L}_{\text{NLL}}(\Theta)$. Therefore, before our work, it was an open question whether 1-bit MC under unknown pre-quantization random noise is possible.

Our Theorems 12-13 provide an affirmative answer to this question. Particularly, under uniformly distributed Λ_k , sub-Gaussian or even heavy-tailed ϵ_k , we formulate 1-bit CS as a convex programming problem and establish theoretical guarantee. Unlike the likelihood approach, we now use a generalized quadratic loss

$$\mathcal{L}(\boldsymbol{\Theta}) = \frac{1}{2n} \sum_{k=1}^{n} \left(\left\langle \boldsymbol{X}_{\boldsymbol{k}}, \boldsymbol{\Theta} \right\rangle - \gamma \cdot \dot{Y}_{k} \right)^{2}.$$
(D.8)

For the core idea behind, while maximum likelihood estimation is a standard estimation strategy, the inspiration of (D.8) is drawn from Lemma 1, i.e., $\gamma \cdot \dot{Y}_k$ can serve as a surrogate of the full observation $Y_k = \langle X_k, \Theta^* \rangle + \epsilon_k$.

At first glance, one may feel that (D.8) is a bit coarse compared to negative log-likelihood, but under uniform dither Λ_k and sub-Gaussian ϵ_k our estimator achieves near minimax rate

⁸This rank surrogate is called max-norm but totally different from $\|.\|_{\text{max}}$ in our work. To avoid confusion, we refer readers to [23] for the details.

(Theorem 12). For comparison, we go back to the noiseless (i.e., $\epsilon_k = 0$ in (D.7)) and exactly low-rank (i.e., q = 0 in (4.3)) case. In this case, Theorem 12 gives a bound $\tilde{O}((\alpha^*)^2 \frac{rd}{n})$ for mean squared error. This is faster than $\tilde{O}((\alpha^*)^2 \sqrt{\frac{rd}{n}})$ obtained in two pioneering works [23,34], and similar to the more recent paper [67].

To conclude, we present the first result for 1-bit MC with unknown pre-quantization random noise, which can either be sub-Gaussian or heavy-tailed. In addition, by some extra technicalities, we believe our method can be extended to both deterministic sparse corruption in [48] and sign flipping noise in [86].

E Details and Algorithms in Experiments

E.1 Sparse Covariance Matrix Estimation

E.1.1 Detailed Simulation

To generate the $d \times d$ underlying covariance matrix Σ^* that satisfies Assumption 1 with q = 0and sparsity s, we first construct

$$\Sigma_0^* = egin{pmatrix} \Sigma_1^* & 0 \ 0 & I_{d-3s} \end{pmatrix},$$

where $\Sigma_1^* = \text{diag}(\Sigma_2^*, \Sigma_2^*, \Sigma_2^*) \in \mathbb{R}^{3s \times 3s}$, and $\Sigma_2^* = [\sigma_{2,ij}^*] \in \mathbb{R}^{s \times s}$ are defined as $\sigma_{2,ii}^* = 1$ for $i \in [s], \sigma_{2,12}^* = \sigma_{2,21}^* = 0.99 - (s-2) \cdot 0.03$, $\sigma_{2,ij}^* = 0.03$ for all other entries. By normalizing the operator norm, we set

$$\Sigma^* = rac{\Sigma^*_0}{\|\Sigma^*_0\|_{\mathrm{op}}}.$$

We i.i.d. draw sub-Gaussian $X_k \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}^*)$, and draw heavy-tailed X_k from Student's t distribution via the Matlab function "mvtrnd(·)" with $\nu = 6$. Then, we apply the 1-bit quantization scheme with parameters slightly tuned to be well-functioning, to obtain the binary data $\{\dot{X}_{kj} : k \in [n], j = 1, 2\}$. Now, we can directly construct the 1-bit estimator $\widehat{\mathbf{\Sigma}}$ defined in (2.4), (2.18), and track the experimental recovery error. In our results, each experiment is obtained as the mean value of 15 independent runs.

E.2 Sparse Linear Regression

E.2.1 Detailed Simulation

We conduct numerical experiments of 1-bit QC-CS (Theorems 7-8) and 1-bit CS (Theorems 9-10). We consider isotropic covariate (i.e., $\mathbb{E}X_k X_k^T = \mathbf{I}_d$), which admits Assumption 3 required for 1-bit QC-CS. For the covariate, sub-Gaussian X_k are generated from Gaussian distribution, while entries of heavy-tailed X_k are i.i.d. drawn from $\sqrt{\frac{2}{3}} \cdot t(\nu = 6)$. Here, $t(\nu = 6)$ represents Student's t distribution with 6 degrees of freedom, and $\sqrt{\frac{2}{3}}$ aims to normalize the variance. We set the first *s* entries of Θ^* to be $\frac{1}{\sqrt{s}}$, while other entries are 0, hence Θ^* is (exactly) *s*-sparse. Sub-Gaussian and heavy-tailed noise ϵ_k are respectively drawn from $\mathcal{N}(0, \sqrt{\frac{3}{5}})$ and $0.3 \cdot t(\nu = 6)$. All these parameters specify the model, so we can generate the full data $\{(X_k, Y_k) : k \in [n]\}$ for a specific (n, d, s).

Then we apply the 1-bit quantization scheme to quantize $\{(X_k, Y_k) : k \in [n]\}$ to $\{(\dot{X}_{k1}, \dot{X}_{k2}, \dot{Y}_k) : k \in [n]\}$ in 1-bit QC-CS, or $\{(X_k, \dot{Y}_k) : k \in [n]\}$ in 1-bit CS. All parameters are properly set according to the Theorems, and we stress that the truncation and dithering parameters for X_k , Y_k are different. For instance, in sub-Gaussian 1-bit QC-CS we use dithering noise $\Lambda_k \sim \text{uni}([-\gamma_Y, \gamma_Y), \Gamma_{kj} \sim \text{uni}([-\gamma_X, \gamma_X]^d))$ with $\gamma_X \neq \gamma_Y$. After the data quantization, we can solve the proposed convex programming problems to obtain the estimator $\widehat{\Theta}$. We track the ℓ_2 norm error $\|\widehat{\Theta} - \Theta^*\|_2$ and report the mean value of 15 independent runs.

E.2.2 Algorithm

Note that the convex programming problems (3.17), (3.26) and (3.30) share the common formulation of

$$\widehat{\Theta} \in \underset{\Theta \in \mathbb{R}^d}{\operatorname{arg\,min}} \quad \frac{1}{2} \Theta^T \widehat{\Sigma}_1 \Theta - \widehat{\Sigma}_2^T \Theta + \lambda \|\Theta\|_1, \tag{E.1}$$

where $\widehat{\Sigma}_1$ is positive semi-definite, $\widehat{\Sigma}_2 \in \mathbb{R}^d$. Here, we use alternating direction method of multipliers (ADMM) to solve (E.1), and the convergence of our algorithm is guaranteed since the variable is divided into two blocks [47]. For more details of ADMM, we refer readers to the survey paper [17].

We now invoke the framework of ADMM and show the iterative formula. Divide $\Theta \in \mathbb{R}^d$ into $M, Z \in \mathbb{R}^d$, (3.17) is equivalent to

$$\underset{M,Z \in \mathbb{R}^d}{\operatorname{arg\,min}} \quad \frac{1}{2} M^T \widehat{\boldsymbol{\Sigma}}_1 M - \widehat{\boldsymbol{\Sigma}}_2^T M + \lambda \|Z\|_1, \text{ s.t. } M = Z.$$

By introducing the multiplier $\Upsilon \in \mathbb{R}^d$, the augmented Lagrangian function reads

$$\frac{1}{2}M^T\widehat{\Sigma}_1M - \widehat{\Sigma}_2^TM + \lambda \|Z\|_1 + \Upsilon^T(M - Z) + \frac{\rho}{2}\|M - Z\|_2^2.$$

Minimizing (M, Z) alternatively and updating Υ via gradient ascent give the iteration formulas

$$\begin{cases} M_{t+1} = (\widehat{\Sigma}_1 + \rho \cdot I_d)^{-1} (\widehat{\Sigma}_2 + \rho \cdot Z_t - \Upsilon_t) \\ Z_{t+1} = \mathcal{S}_{\lambda/\rho} (M_{t+1} + \rho^{-1} \cdot \Upsilon_t) \\ \Upsilon_{t+1} = \Upsilon_t + \rho \cdot (M_{t+1} - Z_{t+1}) \end{cases}$$
(E.2)

that updates (M_t, Z_t, Υ_t) to $(M_{t+1}, Z_{t+1}, \Upsilon_{t+1})$. In (E.2), we define $\mathcal{S}_{\beta}(x) = \operatorname{sign}(x) \max\{0, |x| - \beta\}$ if $x \in \mathbb{R}$, and then let $\mathcal{S}_{\beta}(\cdot)$ element-wisely operate on vectors. This is known as the soft thresholding operator.

E.3 Low-rank Matrix Completion

E.3.1 Detailed Simulation

We simulate low-rank matrix completion with exactly low-rank matrix Θ^* . The $d \times d$ rank r underlying matrix Θ^* is generated by the formulation $\Theta^* = \frac{\Theta_l \Theta_r}{\|\Theta_l \Theta_r\|_{\mathrm{F}}}$, where entries of $\Theta_l \in \mathbb{R}^{d \times r}$ and $\Theta_r \in \mathbb{R}^{r \times d}$ are i.i.d. drawn from $\mathcal{N}(0, 1)$. Furthermore, Θ^* with different

(d, r) are controlled to possess comparable spikiness $\alpha(\Theta^*) = \frac{d\|\Theta^*\|_{\max}}{\|\Theta^*\|_{\mathrm{F}}}$. While the covariate is specified to be $\mathbf{X}_{\mathbf{k}} = e_{k(i)}e_{k(j)}^T$ with $(k(i), k(j)) \sim \mathrm{uni}([d] \times [d])$ (4.2), we test both sub-Gaussian noise and heavy-tailed noise. Specifically, sub-Gaussian or heavy-tailed ϵ_k are i.i.d. copies of $\mathcal{N}(0, \frac{1}{400})$ or $\frac{1}{250} \cdot (\frac{1}{\sqrt{3}}t(\nu=3))$, respectively. Here, $\frac{1}{\sqrt{3}}t(\nu=3)$ is the Student's t distribution with 3 degrees of freedom and variance rescaled to 1. Following these parameters, the full data $\{(\mathbf{X}_k, Y_k) : k \in [n]\}$ are obtained from the model $Y_k = \langle \mathbf{X}_k, \Theta^* \rangle + \epsilon_k$. The responses are processed by the 1-bit quantization scheme and quantized to 1-bit \dot{Y}_k , then solving the convex programming problem (4.6) gives the estimator $\widehat{\Theta}$. In (4.6), we set $\alpha^* = \|\Theta^*\|_{\max}$ and properly tune λ so that it balances the data fidelity and low-rank structure. We track the Frobenius norm error $\|\widehat{\Theta} - \Theta^*\|_{\mathrm{F}}$ and report the mean value of 15 independent trials.

E.3.2 Algorithm

We similarly apply ADMM to solve (4.6), and first separate variable Θ to be two blocks $M, Z \in \mathbb{R}^{d \times d}$. Define $\mathbb{1}'(E)$ to be the indicator function widely used in optimization, i.e., $\mathbb{1}'(E) = 0$ if E happens, $\mathbb{1}'(E) = \infty$ otherwise. Then, we can move the max-norm constraint to objective and obtain the equivalent program

$$\underset{\boldsymbol{M},\boldsymbol{Z}\in\mathbb{R}^{d\times d}}{\operatorname{arg\,min}} \ \frac{1}{2n} \sum_{k=1}^{n} \left(\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{M}\right\rangle - \gamma\cdot\dot{Y}_{k} \right)^{2} + \mathbb{1}' (\|\boldsymbol{M}\|_{\max} \leq \alpha^{*}) + \lambda \|\boldsymbol{Z}\|_{\operatorname{nu}}, \text{ s.t. } \boldsymbol{M} = \boldsymbol{Z}.$$

Let $\boldsymbol{\Upsilon} \in \mathbb{R}^{d \times d}$ be the multiplier, we have the augmented Lagrangian function

$$\frac{1}{2n}\sum_{k=1}^{n}\left(\left\langle \boldsymbol{X}_{\boldsymbol{k}},\boldsymbol{M}\right\rangle-\gamma\cdot\dot{Y}_{k}\right)^{2}+\mathbb{1}'(\|\boldsymbol{M}\|_{\max}\leq\alpha^{*})+\lambda\|\boldsymbol{Z}\|_{\mathrm{nu}}+\left\langle\boldsymbol{\Upsilon},\boldsymbol{M}-\boldsymbol{Z}\right\rangle+\frac{\rho}{2}\|\boldsymbol{M}-\boldsymbol{Z}\|_{\mathrm{F}}^{2}.$$

Some additional notations are necessary before presenting the algorithms. Let $\mathcal{I}_{ij} = \{k \in [n] : \mathbf{X}_k = e_i e_j^T\}$, then we define $\mathbf{J}_1 = [\mathbf{J}_1(i, j)], \mathbf{J}_2 = [\mathbf{J}_2(i, j)] \in \mathbb{R}^{d \times d}$ as

$$\boldsymbol{J_1}(i,j) = \sum_{k \in \mathcal{I}_{ij}} \gamma \cdot \dot{Y}_k , \quad \boldsymbol{J_2}(i,j) = \sum_{k \in \mathcal{I}_{ij}} 1 = |\mathcal{I}_{ij}|.$$

We define $\mathcal{P}_{\Omega}(\cdot)$ to be the projection onto $\Omega \subset \mathbb{R}^{d \times d}$ under Frobenius norm. Let **1** be the all-ones matrix with self-evident size, and \oslash represents the element-wise division between two matrices of the same size. Furthermore, we introduce the soft thresholding operator $\mathcal{S}_{\beta}(\cdot)$ for a matrix \mathbf{A} that admits singular value decomposition $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^*$, where the singular values of \mathbf{A} are arranged in the diagonal matrix Σ . Based on $\mathcal{S}_{\beta}(x) = \operatorname{sign}(x) \max\{0, |x| - \beta\}$ for $x \in \mathbb{R}$, we define $\mathcal{S}_{\beta}(\mathbf{A}) = \mathbf{U} \mathcal{S}_{\beta}(\Sigma) \mathbf{V}^*$ and let $\mathcal{S}_{\beta}(\cdot)$ element-wisely operates on the diagonal matrix Σ . Now, one can derive the ADMM iteration formulas as

$$\begin{cases} \boldsymbol{M}_{t+1} = \mathcal{P}_{\|\boldsymbol{M}\|_{\max} \leq \alpha^*} [(n\rho \cdot \boldsymbol{Z}_t + \boldsymbol{J}_1 - n\boldsymbol{\Upsilon}_t) \oslash (n\rho \cdot \boldsymbol{1} + \boldsymbol{J}_2)] \\ \boldsymbol{Z}_{t+1} = \mathcal{S}_{\lambda/\rho} (\rho^{-1} \cdot \boldsymbol{\Upsilon}_t + \boldsymbol{M}_{t+1}) \\ \boldsymbol{\Upsilon}_{t+1} = \boldsymbol{\Upsilon}_t + \rho \cdot (\boldsymbol{M}_{t+1} - \boldsymbol{Z}_{t+1}) \end{cases}$$
(E.3)